

A NEW ALGORITHM TO CONVERT DECIMAL NUMBER INTO TBNS

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Abstract— Computational complexities arise to convert any decimal number to DBNS are the most important issues to design an algorithm further for the same. Here we have introduced a new algorithm that converts any decimal number to TBNS in a simpler way. Advantages of TBNS over DBNS have been discussed. Although the material presented in this article is mainly theoretical, the proposed algorithm could lead to very efficient implementation for digital signal processing application.

Keywords—DBNS, TBNS[5], DSP[3], OSTROWSKI'S NUMBER SYSTEM[4].

I. INTRODUCTION

The Double-Base number system (DBNS), introduced by V. Dimitrov and G. A. Jullien[1] has advantages in many application like cryptography[2] and digital signal processing (DSP)[3]. R. Muscedere addresses the problem of addition, subtraction and conversion from binary. Valerie Berthe and Laurent Imbert introduce the problem of converting a number from binary to DBNS using continued fraction, Ostrowski's number system, and Diophantine approximation. The main objective of this paper is to convert any decimal number into TBNS which leads great advantages over DBNS.

In Triple Base Number System, we represent any integer in the form

$$x = \sum_i 2^{a_i} 3^{b_i} 5^{c_i} \quad (1)$$

Where, $(a_i, b_i, c_i) > 0$ and $(a_i, b_i, c_i) \neq (a_j, b_j, c_j)$ if $i \neq j$ and i, j are non-negative, independent integers.

In this paper we try to find out 3 non negative integers a, b, c such that $2^a 3^b 5^c \leq x$, and among the solutions to this problem, $2^a 3^b$ is the largest possible value, i.e.

$$2^a 3^b 5^c = \max \left\{ 2^d 3^e 5^f, \text{ such that } (d, e, f) \in \mathbb{N}^2, \text{ and } 2^d 3^e 5^f \leq x \right\} \quad (2)$$

If we let $(a, b, c) \in \mathbb{N}$ be such that $2^a 3^b 5^c \leq x$, our problem can be reformulated as finding 3 non-negative integers a, b, c such that

$$a \log 2 + b \log 3 + c \log 5 \leq \log x \quad (3)$$

and such that, no other integers $d, e, f \geq 0$ give a better left approximation to $\log x$.

Let us define $\alpha = \log_5 2$ and $\beta = \log_5 3$ and $\gamma = \{\log_5 x\} = \log_5 x - [\log_5 x]$ (γ is the fractional part of $\log_5 x$). Then we try to find out the best left approximation to $\log_5 x$ with non-negative integers. If a, b, c are the solution then, for all $(d, e, f) \in \mathbb{N}^2$, with $a \neq d, b \neq e, c \neq f$, we have

$$(d \alpha + e \beta + f) < (a \alpha + b \beta + c) \leq \gamma + [\log_5 x] \quad (4)$$

A graphical interpretation to this problem is to consider the surface Δ of equation $w = -\alpha u - \beta v + \log_5 x$. The solutions are the points with integer coordinates, located in the area defined by the surface Δ and the axes. The best solution is the point which gives the best approximation to $\log_5 x$.

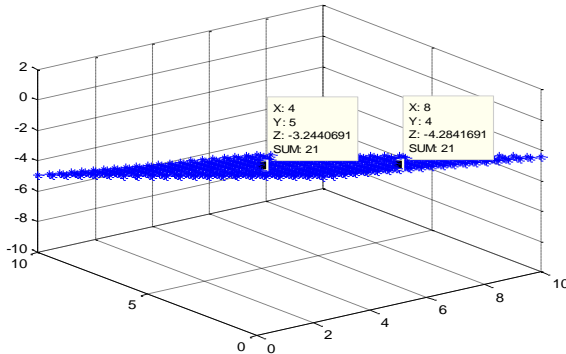


Figure1. Graphical interpretation to the problem of finding the largest 3 integer less than x

II. CONTINUED FRACTIONS AND OSTROWSKI'S NUMBER SYSTEM

A simple continued fraction can be expressed in the form

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Where the partial quotients a_i are integers ≥ 1 . A continued fraction is represented by the sequence $(a_n)_{n \in \mathbb{N}}$ which can either be finite or infinite.

Every irrational real number α can be expressed as an infinite simple continued fraction, and can be written as $\alpha = [a_0, a_1, a_2, a_3, \dots]$. Similarly, every rational number can be expressed as a simple continued fraction. For example, the infinite continued fraction expansions of the irrational e is

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

$$\log_5 2 = [0, 2, 3, 9, 2, 36, 35, \dots]$$

$$\log_5 3 = [0, 1, 2, 6, 1, 1, 1, \dots]$$

The quantity obtained by restricting the continued fractions to its first $(n+1)$ partial quotients

$$p_n / q_n = [a_0, a_1, a_2, a_3, \dots] \quad (5)$$

is called the n th convergent. The series $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$

are computed inductively, starting with $p_{-1}=1, q_{-1}=0, p_0=0, q_0=1$, and for all $n \in \mathbb{N}$

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, q_{n+1} = a_{n+1}q_n + q_{n-1} \quad (6)$$

The sequence of the convergent of an infinite continued fraction gives a series of rational approximations of an irrational number. For example, the convergent of e are listed in TABLE1.

TABLE1. THE FIRST PARTIAL QUOTIENTS AND CONVERGENT OF e

Partial Quotients	Convergent	Value
[2]	2	2.00000000
[2,1]	3	3.00000000
[2,1,2]	8/3	2.66666666
[2,1,2,1]	11/4	2.75000000

III. DEFINITION OF THE SEQUENCE OF NON-HOMOGENEOUS BEST APPROXIMATIONS OF γ

Let the irrationals $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha = [0, a_1, a_2, a_3, \dots]$, $\beta = [0, b_1, b_2, b_3, \dots]$ and $0 < \gamma \leq 1$ be given. Non-homogeneous left approximations of γ are numbers of the form $(j\alpha + k\beta + 1) \leq \gamma$, where j, k, l are integers. It is clear that there is infinitely much such approximation. We are trying to define three increasing sequences of the integers $(j_n)_{n \in \mathbb{N}}, (k_n)_{n \in \mathbb{N}}, (l_n)_{n \in \mathbb{N}}$, such that for all $n \in \mathbb{N}$

$$0 < j_n \alpha - l_n < j_{n+1} \alpha - l_{n+1} < \gamma,$$

$$0 < k_n \beta - l_{2n} < k_{n+1} \beta - l_{2n+1} < \gamma \quad (7)$$

and, furthermore, for all n , for all $j < j_{n+1}, j \neq j_n, k < k_{n+1}, k \neq k_n$ and for all $l \in \mathbb{Z}$ such that $0 \leq j\alpha - l \leq \gamma$, and $0 \leq k\beta - l \leq \gamma$, then

$$0 < j_n \alpha - l_n < j_{n+1} \alpha - l_{n+1} < \gamma, 0 < k_n \beta - l_{2n} < k_{n+1} \beta - l_{2n+1} < \gamma \quad (8)$$

$$\text{For simplicity, } j_n \alpha + k_n \beta - (l_{2n} + l_{1n}) < 2\gamma \quad (9)$$

We, define for all $n, f_n = |\emptyset_n|$. $f_{-1} = 1, f_0 = \alpha, f_1 = 1 - a_1 \alpha$, and for all $n > 1, f_{n+1} = f_{n-1} - a_{n+1} f_n$ (10)

The sequence $(f_n + f_{n+1})_{n \in \mathbb{N}}$ is decreasing, and since $0 < \gamma \leq 1$, there exists a unique non-negative integer n such that $f_n + f_{n+1} < \gamma \leq f_n + f_{n-1}$ (11)

Similarly, for all $n, f_n = |\emptyset_n|$. $f_{-1} = 1, f_0 = \beta, f_1 = 1 - a_1 \beta$, and for all $n > 1, f_{n+1} = f_{n-1} - b_{n+1} f_n$ (12)

The sequence $(f_n + f_{n+1})_{n \in \mathbb{N}}$ is decreasing, and since $0 < \gamma \leq 1$, there exists a unique non-negative

integer n such that $f_n + f_{n+1} < \gamma \leq f_n + f_{n-1}$
 (13)

A. LEMMA1.

Let $0 < \gamma \leq 1$ and $(f_n)_{n \in \mathbb{N}}$ defined in “(10)” “(11)” “(12)” “(13)” then there exists a unique non-negative integer n , a unique non-negative integer c , and a unique real number e such that $\gamma = c f_n + f_{n+1} + e$, where $0 < e \leq f_n$, $1 \leq c \leq a_{n+1}$ or $1 \leq c \leq b_{n+1}$ if $n \geq 1$; and $1 \leq c \leq a_1 - 1$ or $1 \leq c \leq b_1 - 1$ if $n = 0$.

Proof: In $n \geq 1$, then $f_n + f_{n+1} < \gamma \leq f_n + f_{n-1}$ and “(10)”, $f_n < \gamma - f_{n+1} \leq f_n + f_{n-1} - f_{n+1} = (a_{n+1} + 1) f_n$ or $f_n < \gamma - f_{n+1} \leq f_n + f_{n-1} - f_{n+1} = (b_{n+1} + 1) f_n$. If $n = 0$, then $f_0 + f_1 < \gamma \leq f_1 = a_1 f_0 + f_1$ or $f_0 + f_1 < \gamma \leq f_1 = b_1 f_0 + f_1$

B. LEMMA2.

Let the irrationals $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha = [0, a_1, a_2, a_3, \dots]$, $\beta = [0, b_1, b_2, b_3, \dots]$, and $(p_n)_{n \in \mathbb{N}}$, $(q_n)_{n \in \mathbb{N}}$, the sequences of the numerators and denominators of the convergent of α and β . We define the integer's k, l by setting If n is odd $\{k = -c q_n + q_{n+1}, l = -c p_n + p_{n+1}\}$

and if n is even, then

$$\{k = q_n, l = p_n\}$$

Proof: Assume first that n is odd, $\gamma - (j\alpha - l_1)$ or $\gamma - (k\alpha - l_2)$
 $= \gamma - [-c(q_n \alpha - p_n) + (q_{n+1} \alpha - p_{n+1})]$ or $\gamma - [-c(q_n \beta - p_n) + (q_{n+1} \beta - p_{n+1})]$
 $= \gamma - c f_n + f_{n+1} = e$

For n is even $\gamma - (j\alpha - l_1)$ or $\gamma - (k\alpha - l_2) = \gamma - f_n$, and thus $0 < \gamma - (j\alpha - l_1) < \gamma$ or $0 < \gamma - (k\alpha - l_2) < \gamma$.

This algorithm is inspired by [6]. For similar algorithm see [7] [8].

IV. EXPLICIT SOLUTION OF THE NON-HOMOGENEOUS APPROXIMATION PROBLEM

To find out 3 non negative integers a, b, c such that $2^a 3^b 5^c \leq x$, and among the solutions to this problem, $2^a 3^b$ is the largest possible value, i.e.

$$2^a 3^b 5^c = \max \{ 2^d 3^e 5^f, \text{ such that } (d, e, f) \in \mathbb{N}^2, \text{ and } 2^d 3^e 5^f \leq x \}$$

Let $(a, b, c) \in \mathbb{N}$ be one of the solutions to the approximation problem, that is, such that $2^a 3^b 5^c \leq x$, clearly we have

$$a \log 2 + b \log 3 + c \log 5 \leq \log x$$

If $\alpha = \log_5 2$ (α is irrational $0 < \alpha < 1$) and $\beta = \log_5 3$ (β is irrational $0 < \beta < 1$) and $\gamma = \{\log_5 x\} = \log_5 x - [\log_5 x]$ (γ is the fractional part of $\log_5 x$), then the problem

reduces to finding the three non-negative integers a, b, c such that

$$(a \alpha + b \beta + c) \leq \gamma + [\log_5 x], \text{ where } a \leq [\log_2 x], b \leq [\log_3 x], c \leq [\log_5 x].$$

We are thus looking for, $(d, e, f) \in \mathbb{N}^2$ such that

$$\begin{cases} j\alpha + k\beta - l \leq \gamma \\ j\alpha + k\beta - l = \max\{(p\alpha + q\beta - r); \text{ such that } (p, q, r) \in \mathbb{N}\} \\ p \leq [\log_2 x], q \leq [\log_3 x], r \leq [\log_5 x]. \end{cases}$$

From j, k, l we easily get the non- negative exponents a, b, c by setting $a = j, b = k, c = [\log_5 x] - l$.

Example1.

Let $x = 2758178190$. Now, we try to express three non-negative integers' a, b, c such that $2^a 3^b 5^c$ is the largest 3-integer less than or equal to x . let $\alpha = \log_5 2 = 0.4306$, $\beta = \log_5 3 = 0.6826$. We have $\gamma = \{\log_5 x\} = \{13.50647\} = 0.50647$. We set $k_0 = 0, l_0 = 0$. The partial quotients in continued fraction expansion of α, β and corresponding convergent are given in TABLE2.

TABLE2. PARTIAL QUOTIENTS AND CONTINUED FRACTION EXPANSION OF α, β AND THE CORRESPONDING SEQUENCES.

i	0	1	2	3	4
a_i	0	2	3	9	2
p_i	0	1	3	28	59
q_i	1	2	7	65	137
$f_i = \left \frac{q_i \alpha - p_i}{p_i} \right $	0.43067	0.138646	0.014735	0.006023	0.002688
b_i	0	1	2	6	1
p_i	0	1	2	13	15
q_i	1	1	3	19	22
$f_i = \left \frac{q_i \beta - p_i}{p_i} \right $	0.6826	0.3173	0.0478	0.03048	0.01733

TABLE3 and TABLE4 give the first best non-homogeneous left approximations to γ .

TABLE3. BEST LEFT APPROXIMATION OF $\gamma = 0.50647$ WITH NUMBERS OF THE FORM $j\alpha - l_1$

i	e_i	n_i	c_i	j_{i+1}	l_{i+1}	$\gamma - (j_{i+1}\alpha - l_{i+1})$
0	0.50647	1	3	1	0	0.07587
1	0.07587	2	8	8	3	0.06167
2	0.06167	2	7	15	6	0.04747
3	0.04747	2	6	22	9	0.03327
4	0.03327	2	5	29	12	0.01907
5	0.01907	2	4	36	15	0.00487

We get $j = 29, l_1 = (13-12)=1$, here we stop, because the next best left approximation would lead a negative exponent for second base $(13-15)=-2$

TABLE4. BEST LEFT APPROXIMATION OF $\gamma = 0.50647$ WITH NUMBERS OF THE FORM $k\alpha - l_2$

i	e_i	n_i	c_i	k_{i+1}	l_{i+1}	$\gamma - (k_{i+1}\alpha - l_{i+1})$
0	0.50647	1	1	2	1	0.14127
1	0.14127	2	6	5	3	0.09347
2	0.09347	2	5	8	5	0.04567
3	0.04567	4	1	30	20	0.02847

We get $k = 8, l_2 = (13-5) = 8$, here we stop, because the next best left approximation would lead a negative exponent for second base $(13-20=-7)$.

Here, $j\alpha + k\beta + (l_1+l_2) \leq 2\gamma$

Thus, $a = j/2, b = k/2, c = (l_1+l_2)/2$

From this we get $2^{14} 3^5 5^{-4}$.

We apply the same algorithm with the value $(x - 2^{14} 3^5 5^{-4}) = 269858190$. After completion we represent the number as

$$x = 2^{14} 3^5 5^{-4} + 2^{10} 3^1 5^{-7} + 2^9 3^1 5^6 + 2^{11} 3^7 5^0 + 2^{11} 3^0 5^4 + 2^4 3^5 5^2 + 2^3 3^2 5^{-2} + 2^3 3^1 5^2 + 2^6 3^0 5^0$$

V. ADVANTAGES OF TBNS

The basic advantages of TBNS are

- For large values of x, the proposed algorithm is much faster. We have implemented two solutions in TABLE3 and TABLE4. Here the ratios clearly show the efficiency of proposed algorithm.
- TBNS single bit multiplication more advantageous than DBNS single bit multiplication. This may be distinguished in their multiplication unit. For example

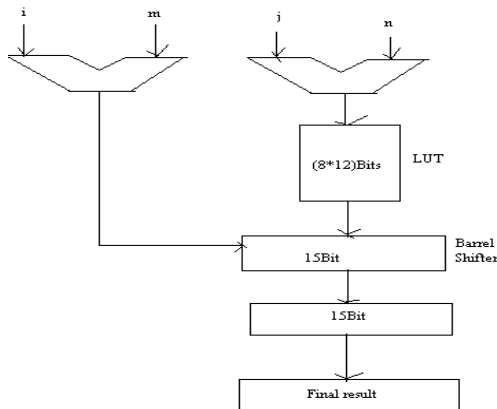


Figure2. Architecture of a DBNS Multiplier Unit (DMU)

First the indices (i, m) and (j, n) of two respective bases are added by binary adders. The result of (j + n) is stored in LUT and then shifted by (i+ m) bits in a single clock using barrel shifter. The final result stored in register. In this way DBNS multiplication is performed.

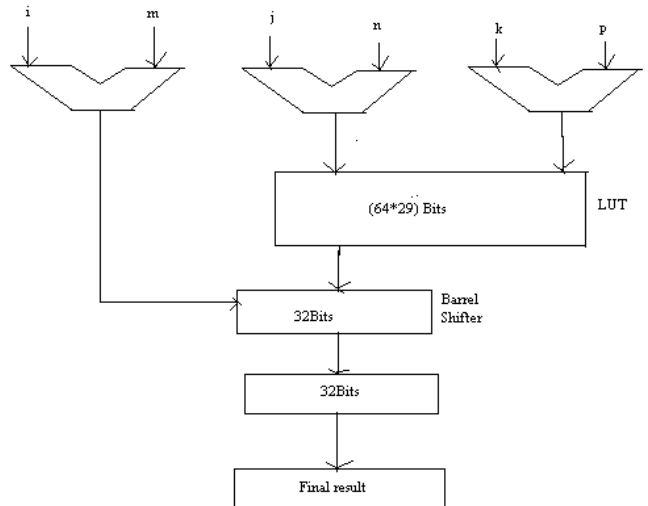


Figure3. Architecture of a TBNS Multiplier Unit (TMU)

First the indices (i, m), (j, n), (k, p) of three respective bases are added by binary adders. The result of (j + n) and (k + p) are stored in LUT and then shifted by (i+ m) bits in a single clock using barrel shifter. The final result stored in register. In this way TBNS multiplication is performed.

- From Figure2 and Figure3 it is clear that for any arithmetic operation the number of hardware required in DBNS is much greater than the number of hardware required in TBNS. In case of multiplication of two numbers the number of hardware required for DBNS and TBNS are shown in TABLE5.

TABLE5. Hardware requirement for DBNS and TBNS

	DBNS Multiplication	TBNS Multiplication
LUT size	(8*12)Bits	(64*29)Bits
Number of multiplication units	25	9
Adder Size	15Bit	32Bit
Number of Adders	24	8

VI. CONCLUSION

A straightforward approach to the problem of finding the largest 3-integer less than or equal to x consists in computing the distance between the surface Δ of equation $w = -\alpha u - \beta v + \log_5 x$ and to keep the values (u, v, w) which lead to the smallest distance. We can consider the plane

$\Delta^1: j = -\log_5^2 k - \log_5^3 l + \log_5 x$ and keep the minimum distance. In Figure4 we have plotted the plane Δ of equation $w = -0.4306u - 0.6826v + 13.50647$, which corresponds to the example1, together with the points we have plotted in Figure4 and those we deduce from proposed algorithm. We clearly remark that the algorithm based on continued fraction and Ostrowski's number system [4].

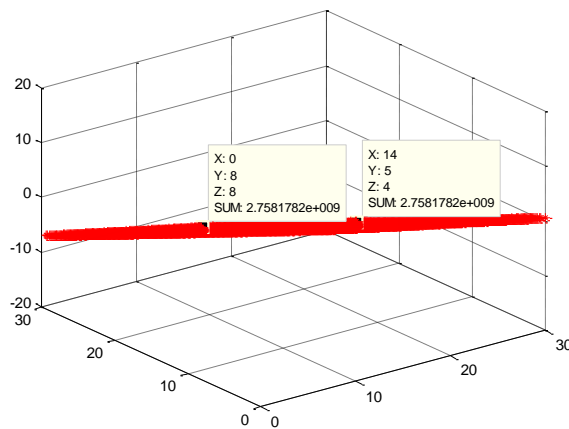


Figure4. Graphical interpretation of the problem of finding the largest 3-integer $\leq x = 2758178190$

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