A SOLUTION OF THE VOLTERRA EQUATION

$$f(x) = g(x) + \int_0^{x^n} k(x,t) f(t) dt$$

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Abstract: In this paper we introduce a series solution for Volterra Equation with limits of integration from 0 to x^n which is a generalization of the classical Volterra equation for $n = 1$.

Keywords: integral equation, Maclaurin series, Leibniz Rule, Functional-differential equations, Volterra equation

I. Introduction

Th volterra integral equation with limits of integral from 0 to x^n, where n is a positive integer is of the form

$$f(x) = g(x) + \int_0^{x^n} k(x,t) f(t) dt$$

where g and k are known functions, and f is the unknown function to be found. As far as the authors know, no solution for the case of n > 1 is given. Actually for n= 1, the equation leads to a differential equation. While for n > 1 the equation leeds to a functional differential equation. We assume here that the functions g, f, and k are analytic at 0 So, we can use the Maclaurian series technique to get our solution, namely

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + ..., \qquad (1)$$

So, our process simply, is to determine the derivatives at 0 of $f(x)$ in terms of the derivatives at 0 of $g(x)$ and $k(x)$ to get the solution of

$$f(x) = g(x) + \int_0^{x^n} k(x,t) f(t) dt \quad , \ n > 1, \qquad (2)$$

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II Solution of the equation

$$f(x) = g(x) + \int_0^{x^n} k(x,t) f(t) dt$$

By Leibniz rule find $f'(x), f''(x), f'''(x),....$

$$f'(x) = g'(x) + \int_0^{x^n} \frac{\partial}{\partial x} k(x,t) f(t) dt + (nx^{n-1}) k(x, x^n) f(x^n)$$

$$= g'(x) + \int_0^{x^n} \frac{\partial}{\partial x} k(x,t) f(t) dt + R_1(x),$$

where

$$R_1(x) = (nx^{n-1}) k(x, x^n) f(x^n),$$

$$R_1(0) = 0 \qquad \text{if} \qquad n > 1$$

$$f''(x) = g''(x) + \int_0^{x^n} \frac{\partial^2}{\partial x^2} k(x,t) f(t) dt + [(2nx^{n-1})k'(x,x^n)$$

$$+ n(n-1)x^{n-2}k(x,x^n)]f(x^n) + (n^2 x^{2n-2})k(x,x^n)f'(x^n)$$

$$= g''(x) + \int_0^{x^n} \frac{\partial^2}{\partial x^2} k(x,t) f(t) dt + R_2(x)$$

Where

$$R_2(x) = [(2nx^{n-1})k'(x,x^n) + n(n-1)x^{n-2}k(x,x^n)]f(x^n)$$
$$+ (n^2 x^{2n-2})k(x,x^n)f'(x^n)$$

$$R_2(0) = 0 \quad \text{if} \quad n > 2$$

$$f'''(x) = g'''(x) + \int_0^{x^n} \frac{\partial^3}{\partial x^3} k(x,t) f(t) dt + [n(n-1)(n-2)x^{n-3}k(x,x^n) + (3n(n-1)x^{n-2})k'(x,x^n)$$

$$+ 3nx^{n-1}k''(x,x^n)]f(x^n) + [(3n^2 x^{2n-2})k'(x,x^n) + (3n^2(n-1))x^{2n-3}k(x,x^n)]f'(x^n)$$

$$+ (n^3 x^{3n-3})k(x,x^n)f''(x^n)$$

$$f'''(x) = g'''(x) + \int_0^{x^n} \frac{\partial^3}{\partial x^3} k(x,t) f(t) dt + R_3(x),$$

Where

$$R_3(x) = [n(n-1)(n-2)x^{n-3}k(x,x^n) + (3n(n-1)x^{n-2})k'(x,x^n)$$

$$+ 3nx^{n-1}k''(x,x^n)]f(x^n) + [(3n^2 x^{2n-2})k'(x,x^n) + (3n^2(n-1))x^{2n-3}k(x,x^n)]f'(x^n)$$

$$+ (n^3 x^{3n-3})k(x,x^n)f''(x^n)$$

$$R_3(0) = 0 \quad \text{if} \quad n > 3$$

$$f^{(4)}(x) = g^4(x) + \int_0^{x^n} \frac{\partial^4}{\partial x^4} k(x,t) f(t) dt + [n(n-1)(n-2)(n-3)x^{n-4})k(x,x^n)$$

$$+ (4n(n-1)(n-2)x^{n-3})k'(x,x^n) + 6n(n-1)x^{n-2}k''(x,x^n) + 4nx^{n-1}k'''(x,x^n)]f(x^n)$$

$$+ [(n^2(n-1)(7n-11)x^{2n-4}k(x,x^n) + (12n^2(n-1)x^{2n-3})k'(x,x^n)$$

$$+ (6n^2 x^{2n-2})k''(x,x^n)]f'(x^n) + [(6n^3(n-1)x^{3n-4})k(x,x^n) + 4n^3 x^{3n-3}k'(x,x^n)]f''(x^n)$$

$$+ [n^4 x^{4n-4}k(x,x^n)]f'''(x^n)$$

$$= g^4(x) + \int_0^{x^n} \frac{\partial^4}{\partial x^4} k(x,t) f(t) dt + R_4(x),$$

where

$$R_4(x) = [n(n-1)(n-2)(n-3)x^{n-4})k(x,x^n) +$$

$$(4n(n-1)(n-2)x^{n-3})k'(x,x^n) + 6n(n-1)x^{n-2}k''(x,x^n) +$$

$$4nx^{n-1}k'''(x,x^n)]f(x^n) + [(n^2(n-1)(7n-11)x^{2n-4}k(x,x^n) +$$

$$(12n^2(n-1)x^{2n-3})k'(x,x^n) + (6n^2 x^{2n-2})k''(x,x^n)]f'(x^n) +$$

$$[(6n^3(n-1)x^{3n-4})k(x,x^n) + 4n^3 x^{3n-3}k'(x,x^n)]f''(x^n) +$$

$$[n^4 x^{4n-4}k(x,x^n)]f'''(x^n)$$

$$R_4(0) = 0 \quad \text{if} \quad n > 4$$

$$f^{(5)}(x) = g^{(5)}(x) + \int_0^{x^n} \frac{\partial^5}{\partial x^5} k(x,t) f(t) dt$$

$$+ [(n(n-1)(n-2)(n-3)(n-4)x^{n-5} k(x,x^n)$$

$$+ (5n(n-1)(n-2)(n-3)x^{n-4}) k'(x,x^n) + 10n(n-1)(n-2)x^{n-3} k''(x,x^n)$$

$$10n(n-1)x^{n-2} k'''(x,x^n) + 5nx^{n-1} k^{(4)}(x,x^n)] f(x^n)$$

$$+ [(5n^2(n-1)(n-2)(3n-5))x^{2n-5} k(x,x^n) + 5n^2(n-1)(7n-11)x^{2n-4}) k'(x,x^n)$$

$$+ (30n^2(n-1)x^{2n-3}) k''(x,x^n) + (10n^2 x^{2n-2}) k'''(x,x^n)] f'(x^n)$$

$$+ [(5n^3(n-1)(5n-7)x^{3n-5}) k(x,x^n) + 30n^3(n-1)x^{3n-4} k'(x,x^n) + 10n^3 x^{3n-3} k''(x,x^n)] f''(x^n)$$

$$+ [10n^4(n-1)x^{4n-5} k(x,x^n) + 5n^4 x^{4n-4} k'(x,x^n)] f'''(x^n) + [n^5 x^{5n-5} k(x,x^n)] f^{(4)}(x)$$

$$= g^{(5)}(x) + \int_0^{x^n} \frac{\partial^5}{\partial x^5} k(x,t) f(t) dt + R_5(x),$$

where

$$R_5(x) = [(n(n-1)(n-2)(n-3)(n-4)x^{n-5} k(x,x^n)$$

$$+ (5n(n-1)(n-2)(n-3)x^{n-4}) k'(x,x^n) + 10n(n-1)(n-2)x^{n-3} k''(x,x^n)$$

$$10n(n-1)x^{n-2} k'''(x,x^n) + 5nx^{n-1} k^{(4)}(x,x^n)] f(x^n)$$

$$+ [(5n^2(n-1)(n-2)(3n-5))x^{2n-5} k(x,x^n) + 5n^2(n-1)(7n-11)x^{2n-4}) k'(x,x^n)$$

$$+ (30n^2(n-1)x^{2n-3}) k''(x,x^n) + (10n^2 x^{2n-2}) k'''(x,x^n)] f'(x^n)$$

$$+ [(5n^3(n-1)(5n-7)x^{3n-5}) k(x,x^n) + 30n^3(n-1)x^{3n-4} k'(x,x^n) + 10n^3 x^{3n-3} k''(x,x^n)] f''(x^n)$$

$$+ [10n^4(n-1)x^{4n-5} k(x,x^n) + 5n^4 x^{4n-4} k'(x,x^n)] f'''(x^n) + [n^5 x^{5n-5} k(x,x^n)] f^{(4)}(x)$$

$$R_5(0) = 0 \quad \text{if} \quad n > 5$$

The general form

$$f^{(m)}(x) = g^{(m)}(x) + \int_0^{x^n} \frac{\partial^m}{\partial x^m} k(x,t) f(t) dt + R_m(x), \tag{3}$$

where,

$$R_m(x) = \sum_{j=1}^{m} n^{m-j} \left(\sum_{c=1}^{j} a_{mjc} p(n, j-c+1) x^{(m-j+1)n-(m-c+1)} k^{(c-1)}(x,x^n) \right) f^{(m-j)}(x^n),$$
$$\tag{4}$$

where

a_{mjc} is real number and $p(n, j-c+) = n(n-1)(n-2)....(n-j+c)$

$R_m(0) = 0$, when $n > m$. thus, replacing x by 0 we obtain

$$f(0) = g(0)$$

$$f'(0) = \begin{cases} g'(0) + R_1(0) & , n = 1 \\ g'(0) & , n > 1 \end{cases}$$

$$f''(0) = \begin{cases} g''(0) + R_2(0) & , n = 2 \\ g''(0) & , n > 2 \end{cases}$$

$$f'''(0) = \begin{cases} g'''(0) + R_3(0) & , n = 3 \\ g'''(0) & , n > 3 \end{cases}$$

$$f^{(4)}(0) = \begin{cases} g^{(4)}(0) + R_4(0) & , n = 4 \\ g^{(4)}(0) & , n > 4 \end{cases}$$

The general form at $x = 0$ is

$$f^{(m)}(0) = \begin{cases} g^{(m)}(0) + R_m(0) & , n = m \\ g^{(m)}(0) & , n > m \end{cases} \tag{5}$$

Applying Maclaurin series

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + ...,$$

So the solution

$$f(x) = \sum_{m=1}^{\infty} (g^{(m)}(0) + R_m(0)) \frac{x^m}{m!} \qquad (6)$$

$$= g(x) + \sum_{m=1}^{\infty} (R_m(0)) \frac{x^m}{m!} \qquad (7)$$

The solution is divided into three types at $n=1$, $n=2$, and $n>2$.

When $n=1$ the solution is in the form

$$f(x) = g(x) + \sum_{m=1}^{\infty} [\sum_{j=1}^{m} a_{mj} k^{(m-j)}(0,0) f^{(j-1)}(0)] \frac{x^m}{m!}, \qquad (8)$$

where a_{mj} is a real number

which is the known solution of Volterra equation ($n=1$).

When $n=2$ the solution is in the form

$$f(x) = g(x) + \sum_{j=2}^{\infty} [k^{(j-2)}(0,0) f(0)] \frac{x^j}{(j-2)!} + \sum_{m=4}^{\infty} [a_m k^{(m-4)}(0,0) f'(0)] \frac{xm}{m!}, \qquad (9)$$

where a_m is a real number

When $n>2$ the solution is in the form

$$f(x) = g(x) + g(0) \sum_{j=n}^{\infty} [k^{(j-n)}(0,0)] \frac{x^j}{(j-n)!} \qquad (10)$$

III. Examples

(1) The solution of the integral equation

$$f(x) = g(x) + \int_{0}^{x^3} k(x,t) f(t) dt ,$$

is

$$f(x) = g(x) + g(0) \sum_{j=3}^{\infty} [k^{(j-n)}(0,0)] \frac{x^j}{(j-3)!}$$

(2) The solution of the integral equation

$$f(x) = 1 - x^3 + \int_{0}^{x^3} (1 - 2t^3 + x^6) f(t) dt,$$

is

$$f(x) = 1$$

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