

Lossless Transmission Lines with Time-Varying Specific Parameters

Vasil G. Angelov

Abstract — The present paper is devoted to the investigation of lossless transmission lines with time-varying specific parameters terminated by nonlinear conductive loads with interval of negative conductance. We give a general method for reducing the mixed problem for the arising hyperbolic system to an initial value problem for neutral system on the boundary. Here we overcome difficulties arising from time-varying specific parameters and formulate conditions for the existence of oscillatory solutions.

Keywords — transmission lines with time-varying specific parameters, oscillatory solutions, nonlinear boundary value problem, fixed point theorem.

I. Introduction

The main goal of the present paper is to consider lossless transmission lines with time-varying specific parameters. They are terminated by a nonlinear conductive load with intervals of negative differential conductance (cf. for instance[1]-[3]).

Consider a lossless transmission line, shown on Fig. 1, terminated by a nonlinear load with $V-I$ characteristic $i = f(u) = \sum_{n=1}^m g_n u^n$, and parallel connected (parasitic) capacitance C_1 , where $E(t)$ is the source function and R_0 – the resistance of the source, $C(x,t)$ – per-unit length capacitance, $L(x,t)$ – per-unit inductance, Λ – the length of the line.

The lossless transmission line is described by the system

$$\frac{\partial i(x,t)}{\partial x} = -\frac{\partial [C(x,t)u(x,t)]}{\partial t}, \quad \frac{\partial u(x,t)}{\partial x} = -\frac{\partial [L(x,t)i(x,t)]}{\partial t}. \quad (1)$$

Under the assumptions (LC):

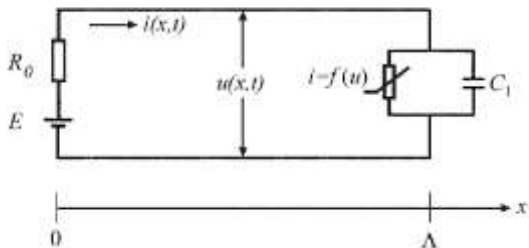


Figure 1. Lossless transmission line with time-varying specific parameters

University of Mining and Geology” St. I. Rilski”
 Department of Mathematics, 1700 Sofia, Bulgaria

$$0 < C_0(1-k) \leq C(x,t) \leq C_0(1+k);$$

$$0 < L_0(1-k) \leq L(x,t) \leq L_0(1+k);$$

$$|\dot{C}_t(x,t)| \leq \dot{C}_0; \quad |\dot{L}_t(x,t)| \leq \dot{L}_0$$

where $C_0 > 0, L_0 > 0, 0 < k < 1, \dot{C}_0 > 0, \dot{L}_0 > 0$ are constants, we formulate a mixed problem for system (1): to find the unknown voltage $u(x,t)$ and current $i(x,t)$ satisfying the system

$$\frac{\partial u(x,t)}{\partial t} + \frac{1}{C(x,t)} \frac{\partial i(x,t)}{\partial x} + \frac{1}{C(x,t)} \frac{\partial C(x,t)}{\partial t} u(x,t) = 0$$

$$\frac{\partial i(x,t)}{\partial t} + \frac{1}{L(x,t)} \frac{\partial u(x,t)}{\partial x} + \frac{1}{L(x,t)} \frac{\partial L(x,t)}{\partial t} i(x,t) = 0$$

for $(x,t) \in \Pi = \{(x,t) \in R^2 : (x,t) \in [0, \Lambda] \times [0, \infty)\}$ with boundary conditions

$$E(t) - u(0,t) = R_0 i(0,t), \quad t \geq 0$$

$$C_1 \frac{du(\Lambda,t)}{dt} = i(\Lambda,t) - \sum_{n=1}^m g_n u^n(\Lambda,t), \quad t \geq 0 \quad (2)$$

and initial conditions

$$u(x,0) = u_0(x), \quad i(x,0) = i_0(x) \quad x \in [0, \Lambda],$$

where $u_0(x), i_0(x)$ are prescribed initial functions.

II. Transformation of the Hyperbolic System in Diagonal Form

The system (1) can be rewritten in the matrix form

$$\begin{bmatrix} u_t \\ i_t \end{bmatrix} + \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} u_x \\ i_x \end{bmatrix} + \begin{bmatrix} C_t/C & 0 \\ 0 & L_t/L \end{bmatrix} \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Via denotations $U = \begin{bmatrix} u \\ i \end{bmatrix}, A = \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}, A_1 = \begin{bmatrix} C_t/C & 0 \\ 0 & L_t/L \end{bmatrix}$

we have

$$U_t + AU_x + A_1U = 0. \quad (3)$$

To transform $A = \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}$ in diagonal form we solve

the equation $\begin{vmatrix} -\lambda & 1/C \\ 1/L & -\lambda \end{vmatrix} = 0$ whose roots are

$$\lambda_1(x,t) = 1/\sqrt{L(x,t)C(x,t)}, \quad \lambda_2(x,t) = -1/\sqrt{L(x,t)C(x,t)}.$$

Eigen-vectors are

$$\left(\xi_1^{(1)}, \xi_2^{(1)}\right) = (\sqrt{C}, \sqrt{L}), \quad \left(\xi_1^{(2)}, \xi_2^{(2)}\right) = (-\sqrt{C}, \sqrt{L}).$$

Denote by H the matrix formed by eigen-vectors

$$H(x,t) = \begin{bmatrix} \sqrt{C(x,t)} & \sqrt{L(x,t)} \\ -\sqrt{C(x,t)} & \sqrt{L(x,t)} \end{bmatrix}. \text{ Its inverse one is}$$

$$H^{-1}(x,t) = \begin{bmatrix} 1/(2\sqrt{C(x,t)}) & -1/(2\sqrt{C(x,t)}) \\ 1/(2\sqrt{L(x,t)}) & 1/(2\sqrt{L(x,t)}) \end{bmatrix} \text{ and then}$$

$$HAH^{-1} = \begin{bmatrix} 1/\sqrt{L(x,t)C(x,t)} & 0 \\ 0 & -1/\sqrt{L(x,t)C(x,t)} \end{bmatrix}.$$

Introduce new variables $Z = \begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix}$, where $Z = HU$ and

$U = H^{-1}Z$, we have

$$V(x,t) = \sqrt{C(x,t)}u(x,t) + \sqrt{L(x,t)}i(x,t)$$

$$I(x,t) = -\sqrt{C(x,t)}u(x,t) + \sqrt{L(x,t)}i(x,t)$$

or

$$u(x,t) = \frac{1}{2\sqrt{C(x,t)}}V(x,t) - \frac{1}{2\sqrt{C(x,t)}}I(x,t)$$

$$i(x,t) = \frac{1}{2\sqrt{L(x,t)}}V(x,t) + \frac{1}{2\sqrt{L(x,t)}}I(x,t).$$

Substituting $U = H^{-1}Z$ in (3) we obtain

$$\frac{\partial(H^{-1}Z)}{\partial t} + A \frac{\partial(H^{-1}Z)}{\partial x} + A_1(H^{-1}Z) = 0$$

and multiplying from the left by H we get

$$\frac{\partial Z}{\partial t} + (HAH^{-1}) \frac{\partial Z}{\partial x} + \left(H \frac{\partial H^{-1}}{\partial t} + HA \frac{\partial H^{-1}}{\partial x} + HA_1 H^{-1} \right) Z = 0.$$

We reach the system

$$\begin{aligned} \frac{\partial V(x,t)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial V(x,t)}{\partial x} + \frac{1}{4} \left(\frac{\partial \ln LC}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial \ln LC}{\partial x} \right) V(x,t) + \\ + \frac{1}{4} \left(\frac{\partial \ln(L/C)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial \ln(L/C)}{\partial x} \right) I(x,t) = 0, \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial I(x,t)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial I(x,t)}{\partial x} + \frac{1}{4} \left(\frac{\partial \ln(L/C)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial \ln(L/C)}{\partial x} \right) V(x,t) + \\ + \frac{1}{4} \left(\frac{\partial \ln LC}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial \ln LC}{\partial x} \right) I(x,t) = 0. \end{aligned}$$

Recall the denotation $v(x,t) = 1/\sqrt{L(x,t)C(x,t)}$ – the speed of propagation. The system

$$\frac{d\xi}{d\tau} = \frac{1}{\sqrt{L(\xi,\tau)C(\xi,\tau)}}, \quad \xi(t) = x$$

$$\frac{d\eta}{d\tau} = -\frac{1}{\sqrt{L(\eta,\tau)C(\eta,\tau)}}, \quad \eta(t) = x$$

for each $(x,t) \in \Pi$ has a unique solution. Then

$$T_\xi(t) = \Lambda/v(\xi(t),t) = \Lambda\sqrt{L(\xi(t),t)C(\xi(t),t)},$$

$$T_\eta(t) = \Lambda/v(\eta(t),t) = \Lambda\sqrt{L(\eta(t),t)C(\eta(t),t)}$$

with derivatives

$$\dot{T}_\xi(t) = \frac{2\Lambda}{4\sqrt{LC}} \left[\frac{\partial(LC)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial(LC)}{\partial x} \right],$$

$$\dot{T}_\eta(t) = \frac{2\Lambda}{4\sqrt{LC}} \left[\frac{\partial(LC)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial(LC)}{\partial x} \right].$$

III. Reducing the Mixed Problem to an Initial Value Problem on the Boundary

For particular case $L = L(t)$, $C = C(t)$ we obtain $\frac{\partial \ln LC}{\partial x} = 0$ and $\frac{\partial \ln(L/C)}{\partial x} = 0$. Consequently

$$T_\xi(t) = \Lambda/v(t) = \Lambda\sqrt{L(t)C(t)} = T_\eta(t) \equiv T(t),$$

$$\dot{T}(t) = \Lambda \left(\dot{L}(t)C(t) + L(t)\dot{C}(t) \right) / (2\sqrt{L(t)C(t)}).$$

We make assumptions (GC):

$$\frac{d^2 \ln(L(t)C(t))}{dt^2} = 0, \quad \frac{d \ln(L(t)/C(t))}{dt} = 0.$$

$$\text{Denote by } \sigma(t) = \frac{1}{4} \frac{d \ln(L(t)C(t))}{dt}, \quad \gamma(t) = \frac{1}{4} \frac{d \ln(L(t)/C(t))}{dt}.$$

Therefore $\dot{\sigma}(t) = 0$ ($\sigma = \text{const}$), $\gamma(t) = 0 \Rightarrow L(t)/C(t) = \text{const}$ and then (4) can be simplified in the form

$$\begin{aligned} \frac{\partial V(x,t)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial V(x,t)}{\partial x} + \sigma V(x,t) = 0 \\ \frac{\partial I(x,t)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial I(x,t)}{\partial x} + \sigma I(x,t) = 0. \end{aligned} \quad (5)$$

Let us set $V(x,t) = e^{-\sigma t} W(x,t)$, $I(x,t) = e^{-\sigma t} J(x,t)$ and substitute in (5). We get

$$\frac{\partial W(x,t)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial W(x,t)}{\partial x} = 0, \quad \frac{\partial J(x,t)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial J(x,t)}{\partial x} = 0. \quad (6)$$

To obtain new boundary conditions with respect to the new variables we substitute

$$u(x,t) = \frac{e^{-\sigma t}}{2\sqrt{C(t)}} W(x,t) - \frac{e^{-\sigma t}}{2\sqrt{C(t)}} J(x,t),$$

$$i(x,t) = \frac{e^{-\sigma t}}{2\sqrt{L(t)}} W(x,t) + \frac{e^{-\sigma t}}{2\sqrt{L(t)}} J(x,t)$$

into (2) and obtain:

$$\begin{aligned} E(t) - \frac{1}{2\sqrt{C(t)}} e^{-\sigma t} W(0,t) + \frac{1}{2\sqrt{C(t)}} e^{-\sigma t} J(0,t) &= \\ = \frac{R_0}{2\sqrt{L(t)}} e^{-\sigma t} W(0,t) + \frac{R_0}{2\sqrt{L(t)}} e^{-\sigma t} J(0,t), \quad t \geq 0, \\ C_1 \frac{d}{dt} \left(\frac{e^{-\sigma t} W(\Lambda,t) - e^{-\sigma t} J(\Lambda,t)}{2\sqrt{C(t)}} \right) &= \frac{e^{-\sigma t} W(\Lambda,t)}{2\sqrt{L(t)}} + \\ + \frac{e^{-\sigma t} J(\Lambda,t)}{2\sqrt{L(t)}} - \sum_{n=1}^m \frac{g_n}{2^n} \left(\frac{e^{-\sigma t} W(\Lambda,t) - e^{-\sigma t} J(\Lambda,t)}{\sqrt{C(t)}} \right)^n, \quad t \geq 0. \end{aligned}$$

For the new initial conditions we proceed from

$$W(x,t) = e^{\sigma t} \sqrt{C} u(x,t) + e^{\sigma t} \sqrt{L} i(x,t),$$

$$J(x,t) = -e^{\sigma t} \sqrt{C} u(x,t) + e^{\sigma t} \sqrt{L} i(x,t).$$

Then

$$W(x,0) = \sqrt{C(0)} u(x,0) + \sqrt{L(0)} i(x,0) =$$

$$= \sqrt{C(0)} u_0(x) + \sqrt{L(0)} i_0(x) \equiv W_0(x)$$

$$J(x,0) = -\sqrt{C(0)} u(x,0) + \sqrt{L(0)} i(x,0) =$$

$$= -\sqrt{C(0)} u_0(x) + \sqrt{L(0)} i_0(x) \equiv J_0(x).$$

So on the base of results from [4] and [5] we formulate the following mixed problem: to solve (6) in $x \in [0, \Lambda]; t \in [0, \infty)$ with initial and boundary conditions

$$W(x,0) = W_0(x), \quad J(x,0) = J_0(x), \quad x \in [0, \Lambda].$$

$$2E(t)e^{\sigma t} - \frac{W(0,t) + J(0,t)}{\sqrt{C(t)}} = R_0 \frac{W(0,t) + J(0,t)}{\sqrt{L(t)}}, \quad t \geq 0$$

$$\frac{d}{dt} \left(\frac{e^{-\sigma t}}{\sqrt{C(t)}} (W(\Lambda,t) - J(\Lambda,t)) \right) - e^{-\sigma t} \frac{W(\Lambda,t) + J(\Lambda,t)}{C_1 \sqrt{L(t)}} =$$

$$= - \sum_{n=1}^m \frac{g_n}{2^{n-1} C_1} \left(e^{-\sigma t} \frac{W(\Lambda,t) - J(\Lambda,t)}{\sqrt{C(t)}} \right)^n, \quad t \geq 0.$$

After obvious transformations the boundary conditions become

$$W(0,t) = \frac{2e^{\sigma t} E(t) \sqrt{L(t)}}{R_0 + Z_0(t)} - J(0,t), \quad t \geq 0,$$

$$\dot{J}(\Lambda,t) = \dot{W}(\Lambda,t) - \left(\sigma + \frac{\dot{C}(t)}{2C(t)} + \frac{\sqrt{C(t)}}{C_1 \sqrt{L(t)}} \right) W(\Lambda,t) +$$

$$+ \left(\sigma + \frac{\dot{C}(t)}{2C(t)} - \frac{\sqrt{C(t)}}{C_1 \sqrt{L(t)}} \right) J(\Lambda,t) + \frac{1}{C_1} \sum_{n=1}^m \frac{g_n e^{-(n-1)\sigma t} (W(\Lambda,t) - J(\Lambda,t))^n}{(2\sqrt{C(t)})^{n-1}}.$$

Repeating reasoning from [4] and [5] we integrate along the characteristics and obtain

$$W(0,t) = W(\Lambda, t - T(t)), \quad J(0, t - T(t)) = J(\Lambda, t).$$

Assuming that $W(\Lambda, t) = W(t)$, $J(0, t) = J(t)$ are unknown functions we obtain the system

$$\begin{aligned} \dot{W}(t) &= \dot{J}(t - T(t)) + \left(\sigma + \frac{\dot{C}(t)}{2C(t)} + \frac{\sqrt{C(t)}}{C_1 \sqrt{L(t)}} \right) W(t) - \\ &- \left(\sigma + \frac{\dot{C}(t)}{2C(t)} - \frac{\sqrt{C(t)}}{C_1 \sqrt{L(t)}} \right) J(t - T(t)) - \\ &- \frac{1}{C_1} \sum_{n=1}^m \frac{g_n e^{-(n-1)\sigma t} (W(t) - J(t - T(t)))^n}{(2\sqrt{C(t)})^{n-1}} \equiv F_W \\ J(t) &= \frac{2e^{\sigma t} E(t) \sqrt{L(t)}}{R_0 + \sqrt{L(t)}/C(t)} - W(t - T(t)), \quad t \geq 0 \end{aligned} \quad (7)$$

with delay $T(t)$. In order to formulate a correct problem we should prescribe initial functions on the interval $[-T(0), 0]$. This can be made by transition of the first initial functions along characteristics of the hyperbolic system (cf. [5]). The obtained functions we denote by $W_0(t)$ and $J_0(t)$.

iv. Existence-Uniqueness of an Oscillatory Continuous Solution

Now we are able to formulate the main problem: to find a solution of (7) with advanced prescribed zeroes on an interval $[0, \infty)$, where $W_0(t)$ and $J_0(t)$ are prescribed oscillating functions on the interval $[-T(0), 0]$.

Let $S_T = \{\tau_l\}_{l=1}^n, n \in \mathbb{N}$ be the set of zeroes of the initial function, that is, $W_0(\tau_l) = 0, J_0(\tau_l) = 0$ such that $\tau_1 = -T(0), \tau_n = 0$. Besides

$$\max\{\tau_{l+1} - \tau_l : l = 0, 1, \dots, n\} \leq \sup\{T(t) : t \in [0, \infty)\} = T_0 < \infty.$$

Let $S = \{t_l\}_{l=0}^{\infty}$ be a strictly increasing sequence of real numbers satisfying the following conditions (C):

(C1) $\lim_{l \rightarrow \infty} t_l = \infty$; (C2) for every l there is $s < l$ such that

$$t_l - T(t_l) = t_s, \quad \text{where } t_s \in S_T \cup S.$$

It follows

$$0 \leq \Delta = \inf\{t_{l+1} - t_l : l = 0, 1, 2, \dots\} \leq \sup\{t_{l+1} - t_l : l = 0, 1, 2, \dots\} = T_0 < \infty$$

Introduce the sets

$$M_W = \left\{ W(\cdot) \in C[0, \infty) : W(t_l) = 0 \text{ and } |W(t)| \leq W_0 e^{\mu(t-t_l)}, t \in [t_l, t_{l+1}] \right\}$$

$$M_J = \left\{ J(\cdot) \in C[0, \infty) : J(t_l) = 0 \text{ and } |J(t)| \leq J_0 e^{\mu(t-t_l)}, t \in [t_l, t_{l+1}] \right\}$$

($l = 0, 1, 2, \dots$), where W_0, J_0, μ are positive constants, and the following families of pseudo-metrics

$$\rho_{\mu}^{(l)}(W, \bar{W}) = \max \left\{ e^{-\mu(t-t_l)} |W(t) - \bar{W}(t)| : t \in [t_l, t_{l+1}] \right\},$$

$$\rho_{\mu}^{(l)}(J, \bar{J}) = \max \left\{ e^{-\mu(t-t_l)} |J(t) - \bar{J}(t)| : t \in [t_l, t_{l+1}] \right\}.$$

The set $M_W \times M_J$ turns out into a complete uniform space (cf.[5]) with respect to the saturated family of pseudo-metrics $\rho_{\mu}^{(l)}(W, J, \bar{W}, \bar{J})$ ($l = 0, 1, 2, \dots$). Using (7) we define the

operator $B = (B_W(W, J), B_J(W, J)) : M_W \times M_J \rightarrow M_W \times M_J$
 by the formulas

$$B_W(W, J)(t) := \int_{t_l}^t F_W(W, J)(s) ds - \frac{t-t_l}{t_{l+1}-t_l} \int_{t_l}^{t_{l+1}} F_W(W, J)(s) ds,$$

$$B_J(W, J)(t) = \frac{2e^{\sigma t} E(t) \sqrt{L(t)}}{R_0 + \sqrt{L(t)/C(t)}} - W(t - T(t)),$$

$t \in [t_l, t_{l+1}] (l = 0, 1, 2, \dots)$.

We call a solution of (6) the solution of the operator equation $(W, J) = (B_W(W, J), B_J(W, J))$.

After some preliminary assertions we reach the main result.

Theorem 1. Let the following conditions be fulfilled:

1) The initial functions $W_0(\cdot), J_0(\cdot) \in C^1[-T(0), 0]$ satisfy

$$|W_0(t)| \leq W_0 e^{\mu(t-\tau_l)}, |J_0(t)| \leq J_0 e^{\mu(t-\tau_l)}, t \in [\tau_l, \tau_{l+1}];$$

2) $E(t_k) = 0; e^{\sigma t} |E(t)| \leq W_0 e^{\mu(t-\tau_l)}, t \in [t_l, t_{l+1}]; (W_0 = J_0)$;

3) Assumptions (LC) and (GC) are valid and

$$|\dot{T}(t)| \leq \frac{\Lambda(1+k)}{2(1-k)} \frac{\dot{L}_0 C_0 + L_0 \dot{C}_0}{\sqrt{L_0 C_0}} = \dot{T}_0 < 1;$$

4) The following inequalities are satisfied

$$\frac{e^{-\mu\Delta} J_0}{1-\dot{T}_0} + e^{\mu T_0} \frac{W_0 + J_0 e^{-\mu\Delta}}{\mu} \left(|\sigma| + \frac{\dot{C}_0}{2C_0(1-\kappa)} + \frac{1}{C_1} \sqrt{\frac{C_0(1+k)}{L_0(1-k)}} + \right.$$

$$\left. + \frac{1}{C_1} \sum_{n=1}^m |g_n| \left(\frac{W_0 + J_0 e^{-\mu\Delta}}{2\sqrt{C_0(1-k)}} \right)^{n-1} \frac{e^{(n-1)\mu T_0} + \dots + 1}{n} \right) \leq J_0;$$

$$\frac{2E_0 \sqrt{L_0(1+k)}}{R_0 + \sqrt{L_0(1-k)/C_0(1+k)}} + W_0 e^{-\mu\Delta} \leq J_0;$$

$$K_W = \frac{e^{-\mu\Delta}}{1-\dot{T}_0} + \frac{e^{\mu T_0} (e^{-\mu\Delta} + 1)}{\mu} \left(|\sigma| + \frac{\dot{C}_0}{2C_0(1-k)} + \frac{1}{C_1} \sqrt{\frac{C_0(1+k)}{L_0(1-k)}} + \right.$$

$$\left. + \sum_{n=1}^m \frac{|g_n| (e^{(n-1)\mu T_0} + \dots + 1)}{C_1} \left(\frac{W_0 + J_0 e^{-\mu\Delta}}{2\sqrt{C_0(1-k)}} \right)^{n-1} \right) < 1;$$

$$K_J = \frac{|\sqrt{L_0(1+k)/(C_0(1-k))} - R_0|}{\sqrt{L_0(1-k)/(C_0(1+k))} + R_0} e^{-\mu\Delta} < 1.$$

Then there exists a unique oscillatory solution of (6), belonging to $M_W \times M_J$.

v. Numerical Example

Consider a transmission line with specific parameters $L(t) = L_0(1+k \cos \theta(t))$, $C(t) = C_0(1+k \cos \theta(t))$. Our purpose here is to find the explicit form of $\theta(t)$ in order obtain a “noise-free signal”.

We check assumption (GC).

Assuming $\frac{\ddot{L}(t)L(t) - \dot{L}^2(t)}{L^2(t)} = \frac{\ddot{C}(t)C(t) - \dot{C}^2(t)}{C^2(t)} = 0$ we get

$$\frac{d^2 \ln(L(t)C(t))}{dt^2} = \frac{d}{dt} \left(\frac{\dot{L}(t)}{L(t)} + \frac{\dot{C}(t)}{C(t)} \right) = \frac{\ddot{L}(t)L(t) - \dot{L}^2(t)}{L^2(t)} + \frac{\ddot{C}(t)C(t) - \dot{C}^2(t)}{C^2(t)} = 0.$$

For $\theta(t)$ we have

$$\ddot{C}(t)C(t) - \dot{C}^2(t) = 0 \Rightarrow C_0(1+k \cos \theta(t))C_0(-k \sin \theta(t)\ddot{\theta}(t) - k \cos \theta(t)\dot{\theta}^2(t)) - C_0^2(-k \sin \theta(t)\dot{\theta}(t))^2 = 0.$$

We put $\dot{\theta} = p(\theta) \Rightarrow \ddot{\theta} = \frac{dp}{d\theta} p$ and $\frac{d\theta}{dt} = K_0 e^{-\int \frac{k+\cos \theta}{\sin \theta(1+k \cos \theta)} d\theta}$.

As usually we put $\tan \frac{\theta}{2} = s \Rightarrow \theta = 2 \arctan s \Rightarrow d\theta = \frac{2}{1+s^2} ds$.

Then

$$\frac{d\theta}{dt} = K_0 e^{\ln \frac{(1-k)\tan^2(\theta/2)+1+k}{\tan(\theta/2)}} = K_0 \frac{(1-k)\tan^2(\theta/2)+1+k}{\tan(\theta/2)}. \quad (8)$$

In view of $\theta_0 = \theta(0); \dot{\theta}_0 = \dot{\theta}(0)$ for $t=0$ we have

$$\dot{\theta}_0 = K_0 \frac{(1-k)\tan^2(\theta_0/2)+1+k}{\tan(\theta_0/2)} \Rightarrow K_0 = \frac{\dot{\theta}_0 \tan(\theta_0/2)}{(1-k)\tan^2(\theta_0/2)+1+k}.$$

It follows

$$\int \frac{\tan(\theta/2)}{(1-k)\tan^2(\theta/2)+1+k} d\theta = K_0 t + K_1;$$

$$-\frac{1}{2k} \int \frac{-k \sin \theta}{1+k \cos \theta} d\theta = K_0 t + K_1 \Rightarrow$$

$$\ln(1+k \cos \theta) = -2kK_0 t - 2kK_1 \Rightarrow K_1 = \frac{\ln(1+k \cos \theta_0)}{-2k};$$

$$\cos \theta(t) = \frac{e^{-2k(K_0 t + K_1)} - 1}{k}; \theta(t) = \arccos \frac{e^{-2k(K_0 t + K_1)} - 1}{k}$$

which is valid for $-1 \leq \frac{e^{-2k(K_0 t + K_1)} - 1}{k} \leq 1$.

The functions $|\dot{L}(t)| \leq L_0 k |\dot{\theta}(t)| \leq L_0 k \ddot{\theta} = \dot{L}_0$ and

$|\dot{C}(t)| \leq C_0 k |\dot{\theta}(t)| \leq C_0 k \ddot{\theta} = \dot{C}_0$ are bounded provided $\dot{\theta}(t)$ is

bounded. In view of the right-hand side of (8) $\dot{\theta}(t)$ is unbounded for $\theta(t) \rightarrow \pi$ and $\theta(t) \rightarrow 0$ that is, $(e^{-2k(K_0 t + K_1)} - 1)/k \rightarrow -1$ and $(e^{-2k(K_0 t + K_1)} - 1)/k \rightarrow 1$. Thus we suppose

$$-1 + \delta \leq \cos \theta(t) = \frac{e^{-2k(K_0 t + K_1)} - 1}{k} \leq 1 - \delta \text{ for sufficiently small}$$

$\delta > 0$. Since

$$\dot{\theta} = K_0 \frac{(1-k)\tan^2(\theta/2)+1+k}{\tan g(\theta/2)} =$$

$$= K_0 \left[(1-k) \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} + (1+k) \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \right]$$

we have to substitute in the last right-hand side $\cos\theta(t)$ by $-1+\delta$ and $1-\delta$, and obtain:

$$\dot{\theta}_1 = K_0 \left[(1-k) \sqrt{\frac{2-\delta}{\delta}} + (1+k) \sqrt{\frac{\delta}{2-\delta}} \right];$$

$$\dot{\theta}_2 = K_0 \left[(1-k) \sqrt{\frac{\delta}{2-\delta}} + (1+k) \sqrt{\frac{2-\delta}{\delta}} \right];$$

$$\ddot{\theta} = \max\{\dot{\theta}_1; \dot{\theta}_2\} = 2|K_0| \frac{1+k(1-\delta)}{\sqrt{2-\delta}\sqrt{\delta}}.$$

In other words if we want to obtain “noise-free” signal we have to choose $\theta(t) = \arccos\left(e^{-2k(K_0 t + K_1)} - 1\right)/k$ and then

$$\sigma = \frac{-k\dot{\theta}_0 \tan(\dot{\theta}_0/2)}{(1-k)\tan^2(\dot{\theta}_0/2)+1+k}.$$

Obviously the sign of σ defines the behavior of the solution amplification or attenuation. The $V-I$ characteristic of the nonlinear element

$$f(u) = g_1 u + g_2 u^2 + g_3 u^3 = 0,001u - 0,5u^2 + (1/3)u^3$$

has an interval of negative differential conductance.

Let us take $W_0 \approx J_0 \approx E_0 \approx 10^{-11}$; $k = 0,01$; $\Lambda = 1m$; $\mu = 5.10^9$; $\delta = 0,01$; $L_0 = 0,2\mu H/m$; $C_0 = 5 pF/m$; $R_0 = 35\Omega$; $C_1 = 8.10^{-11}F$.

Then

$$\sqrt{L_0 C_0} = 10^{-9}; \quad \sqrt{L_0 / C_0} = \sqrt{0,04.10^6} = 200 \Omega;$$

$$(0,99) \times 10^{-9} \leq T(t) = \Lambda \sqrt{L(t)C(t)} \leq (1,01) \times 10^{-9};$$

$$\mu\Lambda = 4,95 \Rightarrow e^{\mu\Lambda} = e^{4,95} = 1,41 \times 10^2;$$

$$\mu T_0 = 5,05; \quad e^{\mu T_0} = e^{5,05} = 1,56.10^2;$$

$$\dot{T}_0 = \frac{1,4.10^{-10} |\dot{\theta}_0 \tan(\theta_0/2)|}{0,99 \tan^2(\theta_0/2) + 1,01} < 1 \text{ for sufficiently small } \theta_0.$$

The inequalities from Theorem 1 become

$$\frac{0,9.10^{-3}}{35+198} + e^{-4,95} \leq 1;$$

$$7.10^{-3} \left/ \left(1 - \frac{1,4.10^{-10} |\dot{\theta}_0 \tan(\theta_0/2)|}{0,99 \tan^2(\theta_0/2) + 1,01} \right) \right. +$$

$$\left. \frac{2,6.10^{-9} |\dot{\theta}_0 \tan(\theta_0/2)|}{0,99 \tan^2(\theta_0/2) + 1,01} + 0,1975 + 0,39 + 0,03443 + 5,265.10^{-5} \leq 1;$$

$$K_W = \frac{|198-35|}{198+35} e^{-4,95} = 0,00504 < 1;$$

$$K_J = 7.10^{-3} \left/ \left(1 - \frac{1,4.10^{-10} |\dot{\theta}_0 \tan(\theta_0/2)|}{0,99 \tan^2(\theta_0/2) + 1,01} \right) \right. +$$

$$\left. \frac{2,6.10^{-9} |\dot{\theta}_0 \tan(\theta_0/2)|}{0,99 \tan^2(\theta_0/2) + 1,01} + 0,197 + 0,39 + 0,138 + 4,7.10^{-5} < 1.$$

We can choose an initial approximation

$$W^{(0)}(t) = W_0 \sin\theta(t), \quad J^{(0)}(t) = J_0 \cos\theta(t).$$

References

- [1] L. A. Blackwell, and K. L. Kotzebue, Semiconductor Diode Parametric Amplifiers. Prentice-Hall, New York, 1961.
- [2] W. H. Louisell, Coupled Mode and Parametric Electronics. John Wiley & Sons. Inc. New York, London, 1960.
- [3] A. Scott, Active and Nonlinear Wave Propagation in Electronics. Wiley Interscience, J. Wiley and Sons, New York, 1970.
- [4] K. L. Cooke, and D. W. Krumme, “Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations”, J. Math. Anal. Appl., vol. 24, pp. 372-387, 1968.
- [5] V. G. Angelov A Method for Analysis of Transmission Lines Terminated by Nonlinear Loads. Nova Science, New York, 2014.
- [6] L. O. Chua, C. A. Desoer, and E. S. Kuh, Linear and Nonlinear Circuits. McGraw Hill Book Company, New York, 1987.

About Author:



Born - 1950 in Sofia; PhD in Mathematics - 1981; Assoc. Professor - 1985; Doctor of Technical Sciences - 2004; Professor - 2007; Head of Department of Mathematics, University of Mining and Geology ”St. I. Rilski” – Sofia, Bulgaria.

Research Subjects: Delay Differential Equations, Fixed Point Theory, Classical Electrodynamics, Transmission Line Theory and Nonlinear Circuit Theory.

Monographs: Fixed Points in Uniform Spaces and Applications, Cluj University Press, Romania, 2009; A Method for Analysis of Transmission Lines Terminated by Nonlinear Loads, Nova Science, New York, 2014.

Awards and Honors: 1) Who’s Who in the World since 1992; 2) Gold Medal of the American Biographical Institute – Ghost of the World Forum University of Cambridge UK – 2010; 3) International Peace Prize, United Cultural Convention, USA – 2012; 4) Scientist of the Year 2015, Summit of Leaders, Oxford Academic Union; 5) Expert of International Achievements Research Centre, Chicago, 2016.