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Analytical applications on the Dunkl-type Fock space $F_k(C^d)$

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***Abstract*— In this work, we establish Heisenberg-type uncertainty principle for the Dunkl-type Fock space $F_k(C^d)$. Next, we give an application of the classical theory of reproducing kernels to the Tikhonov regularization problem for operator $L: F_k(C^d) \to H$, where H be a Hilbert space. Finally, we come up with some results regarding the Tikhonov regularization problem and the Heisenberg-type uncertainty principle for the Dunkl-type Segal-Bargmann transform B_k.**

***Keywords*—component, Dunkl-type Fock spaces, extremal functions, Tikhonov regularization, Heisenberg-type uncertainty principle.**

I- Introduction

In this work, by using the theory of operators on Hilbert spaces [3, 4] we establish Heisenberg-type uncertainty principle for the Dunkl-type Fock space $F_k(C^d)$ [1, 11]. In the Dunkl setting, the Heisenberg-type uncertainty principle for the Dunkl transform D_k [2,5], is studied by Rosler in [7] and by Soltani in [15]. This is an important support for show in this work, the Heisenberg-type uncertainty principle for the Dunkl-type Segal-Bargmann transfor B_k [1, 11]. Next, building on the ideas of Matsuura et al. [6], Saitoh [10], and Soltani [12] , we give an application of the classical theory of reproducing kernels [9] to the Tikhonov regularization problem on the Dunkl-type Fock space $F_k(C^d)$. More precisely, let $L: F_k(C^d) \to H$ be a bounded operator from $F_k(C^d)$ into a Hilbert space H, for all $\lambda \succ 0$, $h \in H$, the infimum $\inf_{f \in F_k(C^d)} \left(\lambda \|f\|^2_{F_k(C^d)} + \|h - Lf\|^2_H\right)$, is attained at one function $f^*_{\lambda,h}$, called the extremal function. As applications, we study the extremal functions associated to the following operators:

- the operator $Lf(w) = (B_k)^{-1} f(w)$,
- the operator $Lf(w) = (D_k)^{-1}(B_k)^{-1} f(w)$,
- and the operator $Lf(w) = f(-iw)$.

In the Dunkl setting, the extremal functions are studied by Soltani in several directions, where L is the Dunkl-type Weierstrass transform [12], L is the Dunkl multiplier operators [13,16] and L is the dual Dunkl-Sonine operator [14].

The contents of the paper are as follows. In Section 2, we recall some properties of the Dunkl-type Fock space $F_k(C^d)$ and we establish Heisenberg-type uncertainty principle for this space. In Section 3, we give an application of the classical theory of reproducing kernels to the Tikhonov regularization problem for bounded operator $L: F_k(C^d) \to H$, where H be a Hilbert space.The last section is devoted to give some results concerning the Tikhonov regularization problem and the Heisenberg-type uncertainty principle for the Dunkl-type Segal-Bargmann transform B_k.

II- Dunkl-type Segal-Bargmann space

The Dunkl operators $T_j(k)$; $j = 1,\ldots,d$, on R^d associated with the positive root system $\Re+$ and the multiplicity function k are given, for a function f of class C^1 on R^d, by

$$T_j(k) f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \Re+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}$$

where $\sigma_\alpha x := x - \dfrac{2\langle \alpha, x\rangle}{|\alpha|^2}\alpha$.

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For $y \in R^d$, the initial problem $T_j(k)u(.,y)(x) = y_j u(x,y)$, $j = 1,\ldots,d$, with $u(0,y) = 1$ admits a unique analytic solution on R^d, which will be denoted by $E_k(x,y)$ and called Dunkl kernel [2]. This kernel has a unique analytic extension to $C^d \times C^d$ (see [8]).

For $z, w \in C^d$, we define $K(z,w) := E_k(z,\overline{w})$.

From the properties of the Dunkl kernel [2, 8], the kernel K is continuous and the function $z \to K(z,.)$ is holomorphic for all $w \in C^d$. Further, $K(z,w) = \overline{K(w,z)}$ and $K(z,w)$ is a positive definite kernel. Then, there exists a Hilbert space $F_k(C^d)$ of holomorphic functions on C^d such that K is its reproducing kernel. The Hilbert space $F_k(C^d)$ contains the C-algebra $P(C^d)$ of polynomial functions on C^d as a dense subspace. In particular, if we denote by $\langle .,. \rangle_{F_k(C^d)}$ the inner product in $F_k(C^d)$, then

$$\langle p,q \rangle_{F_k(C^d)} = p(T(k))\overline{q(\overline{z})}_{z=0}, p,q \in P(C^d),$$

where $p(T(k))$ is the operator formed by replacing z_j by $T_j(k)$ for $j = 1,\ldots,d$. We shall call $F_k(C^d)$ the Dunkl-type Fock space.

III- Heisenberg-type uncertainty principle

We consider the following general result from functional analysis.

Theorem 3.1. (See [3, 4]). Let A and B be self adjoint operators on a Hilbert space H. We define the commutator of A and B to $[A,B] := AB - BA$. Then we have that

$$\|(A-a)f\|_H \|(B-b)f\|_H \geq \frac{1}{2}\left|\langle [A,B]f, f\rangle_H\right|$$,for all $f \in Dom(AB) \cap Dom(BA)$, and all $a,b \in R$ .By applying Theorem 3.1 we establish the Heisenberg-type uncertainty principle for the Dunkl-type Fock space $F_k(C^d)$.

Theorem 3.2. Let $f \in F_k(C^d)$. For all $a,b \in R$, we have

$$\|(T_j(k)+z_j-a)f\|_{F_k(C^d)} \|(T_j(k)-z_j+ib)f\|_{F_k(C^d)} \geq N_k(f)$$

where $N_k(f) = \left| \|f\|^2_{F_k(C^d)} + \langle P_j(k)f, f\rangle_{F_k(C^d)} \right|$,

and $P_j(k)f(z) = \sum_{\alpha \in \Re+} k(\alpha)(\alpha_j)^2 f(\sigma_\alpha z)$.

This uncertainty principle can be written as the following.

Theorem 3.3. Let $f \in F_k(C^d)$. Then

$$M_k^+(f) M_k^-(f) \geq \|f\|^2_{F_k(C^d)} (N_k(f))^2$$

where

$$M_k^{\pm}(f) = \|f\|^2_{F_k(C^d)} \|A_k^{\pm}(f)\|^2_{F_k(C^d)} - \left|\langle A_k^{\pm}(f), f\rangle_{F_k(C^d)}\right|^2$$

$$A_k^{\pm}(f) = (T_j(k) \pm z_j) f.$$

IV- Extremal functions on $F_k(C^d)$

Let $\lambda \succ 0$ and let $L : F_k(C^d) \to H$ be a bounded linear operator from $F_k(C^d)$ into a Hilbert space H. We denote by $\langle f,g \rangle_{L,\lambda}$ the inner product defined on the space $F_k(C^d)$ by

$$\langle f,g \rangle_{L,\lambda} := \lambda \langle f,g \rangle_{F_k(C^d)} + \langle Lf, Lg \rangle_H.$$

By using the theory of extremal function and reproducing kernel of Hilbert space [9,10] we establish the extremal function associated to the operator L on the Dunkl-type Fock space $F_k(C^d)$.

Theorem 4.1. Let $\lambda \succ 0$. The Fock space $(F_k(C^d), \langle .,. \rangle_{L,\lambda})$ possesses a reproducing kernel $K_{L,\lambda}(w,z)$; $w,z \in C^d$, which satisfying the equation $(\lambda I + L^* L) K_{L,\lambda}(w,.) = K(w,.)$ .Moreover the kernel $K_{L,\lambda}(w,z)$ satisfies the following properties

(i) $\|K_{L,\lambda}(w,.)\|_{F_k(C^d)} \leq \dfrac{\sqrt{E_k(w,\overline{w})}}{\lambda}$,

(ii) $\|LK_{L,\lambda}(w,.)\|_H \leq \sqrt{\dfrac{E_k(w,\overline{w})}{\lambda}}$,

(iii) $\|L^* LK_{L,\lambda}(w,.)\|_{F_k(C^d)} \leq \sqrt{E_k(w,\overline{w})}$.

From this theorem we obtain the main result of this section .

Theorem 4.2. For any $h \in H$ and for any $\lambda \succ 0$, there exists a unique function $f^*_{\lambda,h}$, where the infimum

$$\inf_{f \in F_k(C^d)} \left(\lambda \|f\|^2_{F_k(C^d)} + \|h - Lf\|^2_H \right)$$ is attained.

Moreover, the extremal function $f^*_{\lambda,h}$ is given by

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$f^*_{\lambda,h}(w)=\left\langle h, LK_{L,\lambda}(w,.)\right\rangle_H$ and satisfies the inequality $\left|f^*_{\lambda,h}(w)\right|\leq\sqrt{\dfrac{E_k(w,\overline{w})}{\lambda}}\|h\|_H$.

V- Dunkl-type Segal-Bargmann transform

Let ω_k denote the weight function $\omega_k(y):=\prod_{\alpha\in\Re+}\left|\langle\alpha,y\rangle\right|^{2k(\alpha)}$, $y\in R^d$.

Let c_k be the Mehta-type constant given by

$$c_k:=\left(\int_{R^d}e^{-|y|^2}\omega_k(y)dy\right)^{-1}.$$

We denote by μ_k the measure on R^d given by $d\mu_k(y):=c_k\omega_k(y)dy$, and by $L^2_k(R^d)$, the space of measurable functions f on R^d, such that

$$\|f\|_{L^2_k(R^d)}:=\left(\int_{R^d}|f(y)|^2dy\right)^{\frac{1}{2}}\prec\infty.$$

We denote by U_k the kernel given for $z,w\in C^d$, by

$$U_k(z,w):=e^{-\frac{(\langle z,z\rangle+\langle w,w\rangle)}{2}}E_k(\sqrt{2}z,w).$$

The kernel U_k gives rise to an integral transform B_k, which is called Dunkl-type Segal-Bargmann transform on C^d, and defined for f in $L^2_k(R^d)$, by

$$B_k(f)(z):=\int_{R^d}U_k(z,x)f(x)d\mu_k(x),\quad z\in C^d.$$

The transform B_k is an isometric isomorphism of $L^2_k(R^d)$ onto $F_k(C^d)$, and for $f\in L^2_k(R^d)$ we have

$$\|B_k(f)\|_{F_k(C^d)}=\|f\|_{L^2_k(R^d)},$$

$$B_kD_k(f)(w)=B_k(f)(-iw),\quad w\in C^d,$$

where D_k is the Dunkl transform [2,5].

Application 5.1. (i) If $H=L^2_k(R^d)$ and $L=(B_k)^{-1}$, then $L^*=B_k$ and $L^*L=I$. Thus, by Theorem 4.1, we deduce that $K_{L,\lambda}(w,z)=\dfrac{1}{\lambda+1}K(w,z)$, and for $h\in L^2_k(R^d)$, $f^*_{\lambda,h}(w)=\dfrac{1}{\lambda+1}B_k(h)(w)$.

(ii) If $H=F_k(C^d)$ and $Lf(w)=f(-iw)$, then $L^*f(w)=f(iw)$ and $L^*L=I$. Thus, by Theorem 4.1, we deduce that $K_{L,\lambda}(w,z)=\dfrac{1}{\lambda+1}K(w,z)$, and for $h\in F_k(C^d)$, $f^*_{\lambda,h}(w)=\dfrac{1}{\lambda+1}h(iw)$.

Application 5.2. By applying Heisenberg-type uncertainty principle for the Dunkl transform ([15], Theorem 2), we deduce the following Heisenberg-type uncertainty principle for the Dunkl-type Segal-Bargmann transform B_k.

Let $a\succ 0$. There exists $C_a\succ 0$ such that for all $f\in L^2_k(R^d)$ we have

$$\left\|e^{-\frac{|x|^2}{2}}f\right\|_{L^2_k(R^d)}\leq C_a\left\||x|^ae^{-\frac{|x|^2}{2}}f\right\|^{\frac{1}{a+1}}_{L^2_k(R^d)}\left[\sum_{j=1}^d\left\|z_jB_k(f)\right\|^2_{F_k(C^d)}\right]^{\frac{1}{2a+2}}$$

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