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Generalized Fock space for the Airy operator

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Abstract— In this work, we introduce the Fock space $F_\nu(C)$ associated to the Airy operator , and we establish Heisenberg-type uncertainty principle for this space. Next, we give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator $T:F_\nu(C)\to H$, where H be a Hilbert space. Finally, we come up with some results regarding the extremal functions, when T is the difference operator and the Dunkl-difference operator, respectively.

***Keywords*—component, Airy-type Fock space, Tikhonov regularization, Heisenberg-type uncertainty principle.**

I- Introduction

The study of several generalizations of the classical Fock spaces has a long and rich history in many different settings (see for instance [11], [12]). In this work, we will try to generalize Airy-type Fock space, to establish Heisenberg-type uncertainty principle for this space; and to give an application of the theory of reproducing kernels to the Tikhonov regularization on this generalized Fock space.

The generalized Airy operator (or hyper-Bessel operator [14, 15]) is third singular differential operator given by

$$L_\nu := \frac{d^3}{dx^3} + \frac{3\nu}{x}\frac{d^2}{dx^2} - \frac{3\nu}{x^2}\frac{d}{dx},$$

where ν is a nonnegative real number. When $\nu = 0$, this operator becomes the third derivative operator for which some analysis were studied by Widder [18] and for some special value of ν the operator L_ν appeared as a radial part of the generalized Airy equation of a nonlinear diffusion type partial differential equation in R^n. Recently, in a nice and long paper, Cholewinski and Reneke [2] studied and extended, for the operator L_ν, the well known theory related to some singular differential operator of second order for which the literature is extensive.

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Next, Fitouhi et al. [7, 8] established a harmonic analysis related to this operator for examples the eigenfunctions, the generalized translation, the Fourier-Airy transform, the heat equation, the heat polynomials, the transmutation operators, … During the last years, the Airy operator have gained considerable interest in various field of mathematics [3-6], [13-15] and in certain parts of quantum mechanics [1]. The results of this work will be useful when discussing the Fock space associated to this operator.

This space is the background of some applications in this contribution. Especially,

• we study the Airy operator and its adjoint operator on the Airy-type Fock space.

• We establish Heisenberg-type uncertainty principle for the Airy-type Fock space.

• We give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator $T:F_\nu(C)\to H$, where H be a Hilbert space.

• We come up with some results regarding the extremal functions associated to the following operators:

- The difference operator $Tf(z) := \frac{1}{z^3}\big(f(z)-f(0)\big)$.

- And the Dunkl-difference operator $Tf(z) := \frac{1}{2z^3}\big(f(z)-f(-z)\big)$

The contents of the paper are as follows. In Section 2, we introduce the Airy-type Fock space $F_\nu(C)$. In Section 3, we establish Heisenberg-type uncertainty principle for this space. In Section 4, we give an application of the theory of reproducing kernels to the Tikhonov regularization problem for bounded linear operator $T:F_\nu(C)\to H$, where H be a Hilbert space. Next, we come up with some results regarding the Tikhonov regularization problem for the difference operator and the Dunkl-difference operator, respectively.

II- Airy-type Fock space

Let $z\in C$ and $\omega_k = e^{2i\pi\frac{k-1}{3}}$, $k=1,2,3$. A function $u(z)$ is called 3-even if $u(\omega_k z)=u(z)$.

For $\lambda\in C$, the initial problem $L_\nu u(z)=\lambda^3 u(z)$, with $u(0)=1$ and $u^{(k)}(0)=0$, $k=1,2$ admits a unique analytic solution on C, which will be denoted by $G_\nu(\lambda z)$ and expanded in power series as

$$G_\nu(\lambda z)=\sum_{n=0}^{\infty}\frac{(\lambda z)^{3n}}{\alpha_{3n}(\nu)},$$ where

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$$\alpha_{3n}(\nu)=3^{3n}n!\frac{\Gamma\left(n+\frac{1}{3}\right)\Gamma\left(n+\nu+\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\nu+\frac{2}{3}\right)}$$

The function $G_\nu(\lambda z)$ is 3-even and defined as the hypergeometric function (see [2]),

$$G_\nu(\lambda z)={}_0F_2\left[\frac{1}{3},\nu+\frac{2}{3}\middle|\left(\frac{\lambda z}{3}\right)^3\right].$$

In particular $|G_\nu(\lambda z)|\le e^{|\lambda||z|}$ and

$$G_0(\lambda z)=\cos_3(-\lambda z)=\sum_{n=0}^{\infty}\frac{(\lambda z)^{3n}}{(3n)!}.$$

We define the Airy-type Fock space $F_\nu(C)$ as the prehilbertian space of 3-even entire functions $f(z)=\sum_{n=0}^{\infty}a_n z^{3n}$ on C such that

$$\|f\|^2_{F_\nu(C)}=\sum_{n=0}^{\infty}|a_n|^2\alpha_{3n}(\nu)\prec\infty.$$

Let f and g be in $F_\nu(C)$ such that $f(z)=\sum_{n=0}^{\infty}a_n z^{3n}$ and $g(z)=\sum_{n=0}^{\infty}b_n z^{3n}$, the inner product in $F_\nu(C)$ is given by

$$\langle f,g\rangle_{F_\nu(C)}:=\sum_{n=0}^{\infty}a_n\overline{b_n}\alpha_{3n}(\nu).$$

The space $F_\nu(C)$ satisfies the following properties.

(i) The function K_ν given by $K_\nu(w,z)=G_\nu(\overline{w}z)$ is a reproducing kernel for the space $F_\nu(C)$.

(ii) If $f\in F_\nu(C)$, then $|f(z)|\le e^{\frac{|z|^2}{2}}\|f\|_{F_\nu(C)}$, $z\in C$.

(iii) The space $F_\nu(C)$ equipped with the inner product $\langle .,.\rangle_{F_\nu(C)}$ is a Hilbert space; and the set $\left\{\frac{z^{3n}}{\sqrt{\alpha_{3n}(\nu)}}\right\}_{n\in N}$ forms an Hilbert's basis for the space $F_\nu(C)$.

III- Heisenberg-type uncertainty principle

We consider the following lemmas.

Lemma 3.1. Let $L_\nu:F_\nu(C)\to F_\nu(C)$, then its adjoint operator L_ν^* is given by $L_\nu^* f(z)=z^3 f(z)$.

Lemma 3.2. $[L_\nu,L_\nu^*]:=L_\nu L_\nu^*-L_\nu^*L_\nu=(6+9\nu)\mathrm{I}+\mathrm{B}_\nu$, where

$$\mathrm{B}_\nu f(z)=9z^2\frac{d^2}{dz^2}f(z)+18(1+\nu)z\frac{d}{dz}f(z).$$

Lemma 3.3. Let $F_\nu(C)$. Then

$$\left\|L_\nu^* f\right\|^2_{F_\nu(C)}=\left\|L_\nu f\right\|^2_{F_\nu(C)}+(6+9\nu)\|f\|^2_{F_\nu(C)}+\langle f,\mathrm{B}_\nu f\rangle_{F_\nu(C)}.$$

By applying Lemma 3.1, Lemma 3.2, Lemma 3.3 and the following theorem.

Theorem 3.4. (See [9, 10]). Let A and B be self adjoint operators on a Hilbert space H. Then we have

$$\|(A-a)f\|_H\|(B-b)f\|_H\ge\frac{1}{2}\left|\langle[A,B]f,f\rangle_H\right|,$$

for all $f\in Dom(AB)\cap Dom(BA)$, and all $a,b\in R$.

We obtain the following Heisenberg-type uncertainty principle for the Airy-type Fock space $F_\nu(C)$.

Theorem 3.5. Let $f\in F_\nu(C)$. For all $a,b\in R$, we have

$$\left\|(L_\nu+z^3-a)f\right\|_{F_\nu(C)}\left\|(L_\nu-z^3+ib)f\right\|_{F_\nu(C)}\ge\mathrm{A}_\nu f,$$

where

$$\mathrm{A}_\nu f=\left|(6+9\nu)\|f\|^2_{F_\nu(C)}+\langle\mathrm{B}_\nu f,f\rangle_{F_\nu(C)}\right|.$$

This uncertainty principle can be written as the following form.

Theorem 3.6. Let $f\in F_\nu(C)$. Then

$$\Delta_\nu^+(f)\Delta_\nu^-(f)\ge\|f\|^2_{F_\nu(C)}\left(\mathrm{A}_\nu f\right)^2,$$

where

$$\Delta_\nu^\pm(f)=\|f\|^2_{F_\nu(C)}\left\|(L_\nu\pm z^3)f\right\|^2_{F_\nu(C)}-\left|\langle(L_\nu\pm z^3)f,f\rangle_{F_\nu(C)}\right|^2.$$

IV- Extremal functions on $F_\nu(C)$

Let $\lambda\succ 0$ and let $T:F_\nu(C)\to H$ be a bounded linear operator from $F_\nu(C)$ into a Hilbert space H. We denote by $\langle f,g\rangle_{L,\lambda}$ the inner product defined on the space $F_\nu(C)$ by

$$\langle f,g\rangle_{L,\lambda}:=\lambda\langle f,g\rangle_{F_\nu(C)}+\langle Tf,Tg\rangle_H.$$

By using the theory reproducing kernels of Hilbert space and building on the ideas of Saitoh [16,17] we establish the extremal function associated to the operator T on the Airy-type Fock space $F_\nu(C^d)$.

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Theorem 4.1. Let $\lambda \succ 0$. The Fock space $\left(F_\nu(C), \langle .,. \rangle_{T,\lambda}\right)$ possesses a reproducing kernel $K_{T,\lambda}(w,z)$; $w,z \in C$, which satisfying the equation $(\lambda I + T^*T)K_{T,\lambda}(w,.) = K_\nu(w,.)$. Moreover the kernel $K_{T,\lambda}(w,z)$ satisfies the following properties.

(i) $\left\|K_{T,\lambda}(w,.)\right\|_{F_\nu(C)} \le \dfrac{e^{\frac{|w|^2}{2}}}{\lambda}$,

(ii) $\left\|TK_{T,\lambda}(w,.)\right\|_H \le \dfrac{e^{\frac{|w|^2}{2}}}{\sqrt{\lambda}}$,

(iii) $\left\|T^*TK_{T,\lambda}(w,.)\right\|_{F_\nu(C)} \le e^{\frac{|w|^2}{2}}$.

From this theorem we obtain the main result of this section .

Theorem 4.2. For any $h \in H$ and for any $\lambda \succ 0$, there exists a unique function $f^*_{\lambda,h}$, where the infimum $\inf_{f \in F_\nu(C)} \left(\lambda \|f\|^2_{F_\nu(C)} + \|h - Tf\|^2_H\right)$ is attained.

Moreover, the extremal function $f^*_{\lambda,h}$ is given by $f^*_{\lambda,h}(w) = \left\langle h, LK_{T,\lambda}(w,.)\right\rangle_H$ and satisfies the inequality

$$\left|f^*_{\lambda,h}(w)\right| \le \|h\|_H \frac{e^{\frac{|w|^2}{2}}}{\sqrt{\lambda}}.$$

Application 4.3. Let H be the prehilbertian space of 3-even entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^{3n}$ on C such that $\|f\|^2_H = \sum_{n=0}^{\infty} |a_n|^2 \alpha_{3n+3}(\nu) \prec \infty$. The space H is a Hilbert space with Hilbert's basis $\left\{\dfrac{z^{3n}}{\sqrt{\alpha_{3n+3}(\nu)}}\right\}_{n \in N}$ and reproducing kernel $S_\nu(w,z) = \sum_{n=0}^{\infty} \dfrac{(\bar{w}z)^{3n}}{\alpha_{3n+3}(\nu)}$.

(i) Let T be the *difference operator* defined on $F_\nu(C)$ by

$$Tf(z) := \frac{1}{z^3}\left(f(z) - f(0)\right).$$

The operator T maps continuously from $F_\nu(C)$ into H, and we have

$$K_{T,\lambda}(w,z) = \frac{1}{\lambda} + \frac{1}{\lambda+1}\left(G_\nu(\bar{w}z) - 1\right),$$

$$TK_{T,\lambda}(w,.)(z) = \frac{1}{\lambda+1}\frac{G_\nu(\bar{w}z) - 1}{z^3},$$

and $f^*_{\lambda,h}(w) = \dfrac{1}{\lambda+1} w^3 h(w)$.

(ii) Let T be the *Dunkl-difference operator* defined on $F_\nu(C)$ by $Tf(z) := \dfrac{1}{2z^3}\left(f(z) - f(-z)\right)$.

The operator T maps continuously from $F_\nu(C)$ into H, and we have

$$K_{T,\lambda}(w,z) = \frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{(\bar{w}z)^{6n}}{\alpha_{6n}(\nu)} + \frac{1}{\lambda+1}\sum_{n=0}^{\infty} \frac{(\bar{w}z)^{6n+3}}{\alpha_{6n+3}(\nu)},$$

$$TK_{T,\lambda}(w,.)(z) = \frac{1}{\lambda+1}\sum_{n=0}^{\infty} \frac{(\bar{w})^{6n+3} z^{6n}}{\alpha_{6n+3}(\nu)},$$

and $f^*_{\lambda,h}(w) = \dfrac{1}{2(\lambda+1)} w^3 \left(h(w) + h(-w)\right)$.

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