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Numerical implementation a stochastic operational matrix based on triangular functions for solving a stochastic nonlinear differential equation

[Zahra Sadati]

Abstract*— In this paper, an efficient method to determine a numerical solution of a stochastic nonlinear differential equation (SNDE) is proposed. The method is illustrated by using a stochastic operational matrix based on the triangular functions (TFs). In this proposed method, the SNDE is reduced to a stochastic nonlinear system of equations and unknowns. Then, the error analysis is demonstrated by using some preliminaries. Finally, the method is evaluated with some numerical examples.. *(Abstract)

Keywords*— *Triangular functions*, Standard Brownian motion, Stochastic operational matrix, Stochastic nonlinear differential equations, Collocation method *(key words)

I. Introduction

Different fields of problems in physics, mechanics, economics, sociology, biological lead to the SNDE that are dependent on a noise source, on a Gaussian white noise. In the last years, many methods are proposed and applied for approximate solutions of the stochastic differential equations (see [5, 8, 9, 10, 13]), because most the stochastic differential equations can not be solved analytically and hence it is of great importance to provide numerical solution. In the presented method we use from the stochastic operational matrix based on properties of the TFs in combination with the collocation technique, because equation is reduced to the stochastic nonlinear system with better accuracy in comparison with other methods.

In paper, we consider

$$\begin{cases} dX(s) = g(s, X(s))ds + h(s, X(s))dB(s), \quad s \in (0,T), \\ X(0) = X_0, \end{cases} \tag{1}$$

or,

$$X(t) = X_0 + \int_0^t g(s, X(s))ds + \int_0^t h(s, X(s))dB(s), \quad t \in (0,T), T < 1, \tag{2}$$

where $g(s, X(s)), h(s, X(s)) : (0,T) \times \mathsf{R} \to \mathsf{R}$ and $X(t)$ are the stochastic processes and unknown on probability space $(\Omega, \hat{\mathsf{O}}, P)$. Also, $B(s)$ is the standard Brownian motion (SBM) defined on probability space.

Note that Eq. (2) play an important role in many application in mathematical finance, biology, medical, social, etc ([2]).

Rest of paper is organized as follows: In Section 2, the first we state some essential preliminaries and conditions of existence a unique solution for Eq. (2), then we introduce the TFs by using properties of the block pulse functions (BFs). In Section 3, properties of the TFs is applied for solving Eq. (2), so, equation is reduced to the stochastic nonlinear system. In Section 4, the proposed method is illustrated by some numerical examples. Finally, in Section 6, provides the conclusion.

II. Preliminaries

Let $\{B(t), t > 0\}$ be the SBM with properties as following:

$B(t) - B(s)$, for $t > s,$ is independent of the past (Independence of increments).

$B(t) - B(s)$ has Normal distribution with mean 0 and variance $t - s$, i.e. $B(t) - B(0)$ has $N(0,t)$ distribution (Normal increments).

$B(t), t \geq 0$ are continuous functions (Continuity of paths).

Definition 2.1 *Let* $\nu = \nu(S,T)$ *be the class of functions* $g(t,\omega) : [0,\infty) \times \Omega \to R$ *such that:*

$(t,\omega) \to g(t,\omega)$, be $B \times \mathsf{F}$-measurable, where B denotes the Borel σ-algebra on $[0,\infty)$ and F be the σ-algebra on Ω.

$g(t,\omega)$ is F_t-adapted, where F_t is the σ-algebra generated by the random variables $B(s)$, for $s \leq t$.

$$E[\int_S^T g^2(t,\omega)dt] < \infty .$$

Theorem 2.2 *(The Itô isometry). Let* $f \in \nu(S,T)$ *, then*

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$$E[(\int_S^T f(t,\omega)dB(t)(\omega))^2] = E[\int_S^T f^2(t,\omega)dt].$$

Now, we state some essential condition on the coefficients of Eq. (2) as following:

A_1. Let the function $g(t,X)$ hold in Lipschitz conditions and Linear growth, i.e. there are constants $k_1, k_2 > 0$ such that:

$$\begin{cases} |\, g(t,X) - g(t,Y)\,| \le k_1 \,|\, X - Y\,|, \ (Lipschitz\ continuity), \\ |\, g(t,X)\,| < k_2(1+|\, X\,|), \ (lineargrowth). \end{cases}$$

A_2. Let the function $h(t,X)$ hold in Lipschitz conditions and Linear growth, i.e. there are constants k_3, k_4 such that:

$$\begin{cases} |\, h(t,X) - h(t,Y)\,| < k_3 \,|\, X - Y\,|, \ (Lipschitz\ continuity), \\ |\, h(t,X)\,| < k_4(1+|\, X\,|), \ (lineargrowth). \end{cases}$$

For x, $y \in R$ and $t \in (0,T)$.

Theorem 2.3 *Let* $g(t,X(t))$ *and* $h(t,X(t))$ *hold in conditions* A_1, A_2 *and* $E\,|\,X_0\,|^2 < \infty$, *then, there exists a unique solution for Eq. (2).*

Proof. See [2, 12].

Finally, we introduce some essential properties of the TFs that are needful for this paper. For more details see [3, 4, 6, 7].

1 . In [1] is defined

$$T_i^1(t) = \begin{cases} 1 - \dfrac{t-ih}{h} & ih \le t < (i+1))h, \\ 0 & elsewhere, \end{cases}$$

and

$$T_i^2(t) = \begin{cases} \dfrac{t-ih}{h} & ih \le t < (i+1))h, \\ 0 & elsewhere, \end{cases}$$

where $h = \dfrac{T}{m}$.

$$T_i^1(t) + T_i^2(t) = \phi_i(t), \text{ where}$$

$$\phi_i(t) = \begin{cases} 1 & (i)h \le t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

is the i th block pulse function and $\Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_j(x), \ldots, \Phi_m(x))^T$ denotes the BPFs .

$T(t).T^T(t) \approx diag(T(t)) = \hat{T}(t)$, where $T(t)$ is called the 1D-TF vector as following:

$$T(t) = [T1(t), T2(t)],$$

with

$$T1(t) = [T_0^1(t), \ldots, T_{m-1}^1(t)]^T,$$

and

$$T2(t) = [T_0^2(t), \ldots, T_{m-1}^2(t)]^T.$$

2 . Let the function $f(x) \in L^2([0,T))$, then

$$f(t) \approx \sum_{i=0}^{m-1} F1_i T_i^1(t) + \sum_{i=0}^{m-1} F2_i T_i^2(t) = F1^T.T1(t) + F2^T.T2(t)$$

$$= [F1, F2]^T.[T1(t), T2(t)] = F^T.T(t),$$

where the vector F is called the 1D-TF coefficient vector, $F1 = (F1_i)_{m\times 1} = (f(ih))_{m\times 1}$

and

$$F2 = (F2_i)_{m\times 1} = (f(i+1)h)_{m\times 1} \ (i = 0,1,..,m-1).$$

3. $\int_0^t T(s)ds \approx P_T.T(t)$, where

$$P_T = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix},$$

with

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}_{m\times m},$$

and

$$P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{m\times m}.$$

4. In [11] is proved that

$$\int_0^t T1(s)dB(s) \approx P1_s.T1(t) + P1_s.T2(t),$$

and

$$\int_0^t T2(s)dB(s) \approx P2_S.T1(t) + P2_S.T2(t),$$

where

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$$P1_S = \begin{pmatrix} \alpha(0) & \beta(0) & \beta(0) & \dots & \beta(0) \\ 0 & \alpha(1) & \beta(1) & \dots & \beta(1) \\ 0 & 0 & \alpha(2) & \dots & \beta(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \beta(m-2) \\ 0 & 0 & 0 & \dots & \alpha(m-1) \end{pmatrix}_{m\times m},$$

and

$$P2_S = \begin{pmatrix} \gamma(0) & \rho(0) & \rho(0) & \dots & \rho(0) \\ 0 & \gamma(1) & \rho(1) & \dots & \rho(1) \\ 0 & 0 & \gamma(2) & \dots & \rho(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \rho(m-2) \\ 0 & 0 & 0 & \dots & \gamma(m-1) \end{pmatrix}_{m\times m},$$

with

$$\begin{cases} \alpha(i) = (i+1)[B((i+0.5)h) - B(ih)] - \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s), \\ \beta(i) = (i+1)[B((i+1)h) - B(ih)] - \int_{ih}^{(i+1)h} \frac{s}{h} dB(s), \\ \gamma(i) = -i[B((i+0.5)h) - B(ih)] + \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s), \\ \rho(i) = -i[B((i+1)h) - B(ih)] + \int_{ih}^{(i+1)h} \frac{s}{h} dB(s). \end{cases}$$

5. In [11] is proved that

$$\int_0^t T(s)dB(s) \approx P_S(T1+T2) = P_S.T(t),$$

where

$$P_S = \begin{pmatrix} P1_S & P1_S \\ P2_S & P2_S \end{pmatrix}.$$

III. Solving the SNDE via the TFs

Let

$$\begin{cases} p(s) = g(s, X(s)), \\ q(s) = h(s, X(s)), \end{cases} \quad (4)$$

with substituting (4) in Eq. (2), we get

$$X(t) = X_0 + \int_0^t p(s)ds + \int_0^t q(s)dB(s). \quad (5)$$

By using properties of the TFs, we can write

$$\begin{cases} p(s) \approx P^T.T(s), \\ q(s) \approx Q^T.T(s), \end{cases} \quad (6)$$

where

$$P = (P_i)_{2m\times 1} = (p(0), p(h), \dots, p((m-1)h), p(h), p(2h), \dots, p(mh))_{2m\times 1},$$

and

$$Q = (Q_i)_{2m\times 1} = (q(0), q(h), \dots, q((m-1)h), q(h), q(2h), \dots, q(mh))_{2m\times 1}.$$

With substituting (6) in (5), we get

$$X(t) \approx X_0 + \int_0^t P^T.T(s)ds + \int_0^t Q^T.T(s)dB(s), \quad (7)$$

or,

$$X(t) \approx X_0 + P^T P_T T(t) + Q^T P_s T(t). \quad (8)$$

Also, by substituting (8) in (4), we obtain

$$\begin{cases} p(t) = g(t, X_0 + P^T P_T T(t) + Q^T P_s T(t)), \\ q(t) = h(t, X_0 + P^T P_T T(t) + Q^T P_s T(t)). \end{cases} \quad (9)$$

Now, with replacing $\approx$ by $=$, the relation (9) is approximated via the collocation method in $m+1$ nodes $t_j = \frac{j}{\frac{1}{T}m+1}$ $(j = 0,1,\dots,m)$, as follows:

$$\begin{cases} p(t_j) = g(t_j, X_0 + P^T P_T T(t_j) + Q^T P_s T(t_j)), \\ q(t_j) = h(t_j, X_0 + P^T P_T T(t_j) + Q^T P_s T(t_j)), \end{cases} \quad (10)$$

or,

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$$\begin{cases} P^T T(t_j) = g(t_j, X_0 + P^T P_T T(t_j) + Q^T P_s T(t_j)), \\ Q^T T(t_j) = h(t_j, X_0 + P^T P_T T(t_j) + Q^T P_s T(t_j)), \end{cases} \quad (11)$$

where (11) is the nonlinear system of $2m+2$ equations and $2m+2$ unknowns. After solving Eq. (11), we conclude that

$$X(t) = X_m(t) = X_0 + P^T P_T T(t) + Q^T P_s T(t). \quad (12)$$

Numerical examples

Example 1 *Let us consider the SNDE as*

$$\begin{cases} dX(s) = \frac{1}{1000} s^3 X(s) ds - \frac{1}{20} s^3 X(s) dB(s), s \in (0,T), T < 1, \\ X(0) = \frac{-1}{50}, \end{cases}$$

with the exact solution $X(t) = \frac{-1}{50} \exp\left(\frac{1}{4000} t^4 - \frac{1}{2800} t^7 - \frac{1}{20} \int_0^t s^3 dB(s)\right)$.

The numerical results have been shown in Table (1), where $\bar{x}$ and $\bar{s}$ are error mean and standard deviation of error, respectively.

Example 2 *Let us consider the SNDE as*

$$\begin{cases} dX(s) = \left(\frac{3}{2} \sqrt[3]{X^2(s)} + \frac{3}{256} \sqrt[3]{X(s)}\right) ds - \\ \frac{3}{16} \sqrt[3]{X^2(s)} dB(s), s \in (0,T), T < 1, \\ X(0) = 0, \end{cases}$$

with the exact solution $X(t) = \frac{1}{8}\left(t - \frac{1}{8} B(t)\right)^3$. The numerical results have been shown in Table (2), where $\bar{x}$ and $\bar{s}$ are error mean and standard deviation of error, respectively.

TABLE I.

t	$\bar{x}_E$	s_E	%95 confidence interval for mean	
			Lower	Upper
0.05	8.676300×10^{-5}	3.237015×10^{-5}	7.2576152×10^{-5}	1.00949×10^{-4}
0.1	4.353135×10^{-4}	7.585946×10^{-5}	4.0206662×10^{-4}	4.6856037×10^{-4}
0.15	1.444730×10^{-3}	2.427305×10^{-4}	1.3383486×10^{-3}	1.55111×10^{-3}
0.2	3.122060×10^{-3}	2.095779×10^{-4}	3.0302084×10^{-3}	3.213911×10^{-3}

TABLE. II

t	$\bar{x}_E$	s_E	%95 confidence interval for mean	
			Lower	Upper
0.05	8.676300×10^{-5}	3.237015×10^{-5}	7.2576152×10^{-5}	1.00949×10^{-4}
0.1	4.353135×10^{-4}	7.585946×10^{-5}	4.0206662×10^{-4}	4.6856037×10^{-4}
0.15	1.444730×10^{-3}	2.427305×10^{-4}	1.3383486×10^{-3}	1.55111×10^{-3}
0.2	3.122060×10^{-3}	2.095779×10^{-4}	3.0302084×10^{-3}	3.213911×10^{-3}

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