On the Existence of n-Tuple Magic Rectangles

Phaisatcha Inpoonjai and Thiradet Jiarasuksakun

Abstract—**Magic rectangles are a classical generalization of the well-known magic squares, and they are related to graphs. A graph G is called degree-magic if there is a labelling of the edges by integers $1, 2, ..., |E(G)|$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1+|E(G)|)\deg(v)/2$. In this paper we generalize magic rectangles to be n-tuple magic rectangles, and prove the necessary and sufficient conditions for the existence of even n-tuple magic rectangles. Using this existence we identify the sufficient condition for degree-magic labellings of the n-fold self-union of complete bipartite graphs to exist.**

Keywords—**magic squares, magic rectangles, degree-magic graphs**

I. Introduction

Magic rectangles are a natural generalization of the magic squares which have widely intrigued mathematicians and the general public. A magic (p, q)-rectangle R is a $p \times q$ array in which the first pq positive integers are placed such that the sum over each row of R is constant and the sum over each column of R is another (different if $p \neq q$) constant. Harmuth [1, 2] studied magic rectangles over a century ago and proved that

Theorem 1 ([1, 2]) *For $p, q > 1$, there is a magic (p, q)-rectangle R if and only if $p \equiv q \pmod 2$ and $(p, q) \neq (2, 2)$.*

In 1990, Sun [3] studied the existence of magic rectangles. Later, Bier and Rogers [4] studied balanced magic rectangles, and Bier and Kleinschmidt [5] studied centrally symmetric and magic rectangles. Then Hagedorn [6] presented a simplified modern proof of the necessary and sufficient conditions for a magic rectangle to exist. The concept of magic rectangles was generalized to n-dimensions and several existence theorems were proven by Hagedorn [7].

For simple graphs without isolated vertices, if G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

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$$f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every} \quad v \in V(G), \quad (1)$$

where $\eta(v, e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all} \quad v \in V(G). \quad (2)$$

A magic labelling f of a graph G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. A graph G is *supermagic* (*magic*) whenever a supermagic (magic) labelling of G exists.

A bijective mapping f from $E(G)$ into $\{1, 2, ..., |E(G)|\}$ is called a *degree-magic labelling* (or only *d-magic labelling*) of a graph G if its index mapping f^* satisfies

$$f^*(v) = \frac{1+|E(G)|}{2}\deg(v) \quad \text{for all} \quad v \in V(G). \quad (3)$$

A d-magic labelling f of G is called *balanced* if for all $v \in V(G)$, the following equation is satisfied

$$\left|\{e \in E(G) : \eta(v, e) = 1, \ f(e) \leq \lfloor |E(G)|/2 \rfloor\}\right|$$
$$= \left|\{e \in E(G) : \eta(v, e) = 1, \ f(e) > \lfloor |E(G)|/2 \rfloor\}\right|. \quad (4)$$

A graph G is *degree-magic* (*balanced degree-magic*) or only *d-magic* when a d-magic (balanced d-magic) labelling of G exists.

The concept of magic graphs was introduced by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. There are now many papers published on magic and supermagic graphs; we refer the reader to Gallian [10] for more comprehensive references. Recently, the concept of degree-magic graphs was introduced by Bezegová and Ivančo [11] as an extension of supermagic regular graphs. They also established the basic properties of degree-magic graphs and proved that

Proposition 1 ([11]) *For $p, q > 1$, the complete bipartite graph $K_{p,q}$ is d-magic if and only if $p \equiv q \pmod 2$ and $(p, q) \neq (2, 2)$.*

Theorem 2 ([11]) *The complete bipartite graph $K_{p,q}$ is balanced d-magic if and only if the following statements hold:*
(i) $p \equiv q \equiv 0 \pmod 2$;
(ii) if $p \equiv q \equiv 2 \pmod 4$, then $\min\{p, q\} \geq 6$.

In this paper we introduce n-tuple magic rectangles. To show their existence, we introduce the closely related concept

of centrally n-tuple symmetric rectangles. Then we use the existence of centrally n-tuple symmetric rectangles to give a construction of even n-tuple magic rectangles. Finally, we identify the sufficient condition for d-magic labellings of the n-fold self-union of complete bipartite graphs to exist.

II. The n-Tuple Magic Rectangles

In this section we introduce n-tuple magic rectangles and prove the necessary and sufficient conditions for even n-tuple magic rectangles to exist.

Definition 1 An n-tuple magic (p, q)-rectangle $R := (r_{i,j}^1)$ $(r_{i,j}^2)...(r_{i,j}^n)$ is a class of n arrays in which each array has p rows and q columns, and the first npq positive integers are placed such that the sum over each row of any array of R is constant and the sum over each column of R is another (different if $p \neq q$) constant.

Let $R := (r_{i,j}^1)(r_{i,j}^2)...(r_{i,j}^n)$ be an n-tuple magic (p, q)-rectangle. As each row sum of any array of R is $q(npq+1)/2$ and each column sum of R is $p(npq+1)/2$ and both are integer, we then have

Proposition 2 If R is an n-tuple magic (p, q)-rectangle, then the following statements hold:
(i) if n is odd, then $p \equiv q$ (mod 2);
(ii) if n is even, then $p \equiv q \equiv 0$ (mod 2).

Proposition 2 allows the set of n-tuple magic rectangles to be divided into sets of odd and even rectangles. We quickly see that an n-tuple magic $(2, 2)$-rectangle does not exist. To show the existence of other even n-tuple magic rectangles, we introduce the closely related concept of centrally n-tuple symmetric (p, q)-rectangles as follows.

Definition 2 Let $x > -1$ and let R be a class of n even rectangular arrays in which each array has p rows and q columns and the entries of R are numbers $\pm(x+1)$, ..., $\pm(x+npq/2)$. R is a *centrally n-tuple symmetric (p, q)-rectangle of type x* if the sum over each row and column of any array is zero. Additionally, if R has an equal number of positive and negative numbers in each row and column of any array, we say that R is *balanced*.

If R is an even n-tuple magic (p, q)-rectangle, then by subtracting $(npq+1)/2$ from each entry of R, we obtain a centrally n-tuple symmetric (p, q)-rectangle of type $-1/2$. Similarly, every centrally n-tuple symmetric (p, q)-rectangle of type $-1/2$ determines an even n-tuple magic (p, q)-rectangle. Thus, we can use the existence of centrally n-tuple symmetric (p, q)-rectangles to prove the existence of even n-tuple magic (p, q)-rectangles.

Lemma 1 For x, $y > -1$, if a balanced centrally n-tuple symmetric (p, q)-rectangle of type x exists, then a balanced centrally n-tuple symmetric (p, q)-rectangle of type y exists.

Proof. Suppose that $R := (r_{i,j}^1)(r_{i,j}^2)...(r_{i,j}^n)$ is the given rectangle. Then we define a (p, q)-rectangle $S := (s_{i,j}^1)(s_{i,j}^2)$ $...(s_{i,j}^n)$ by

$$s_{i,j}^t = (y-x)\mathrm{sgn}(r_{i,j}^t) + r_{i,j}^t, \quad \text{for every} \quad t \in \{1, 2, ..., n\}.$$

The entries of S are the numbers $\pm(y+1)$, ..., $\pm(y+npq/2)$. For any $t \in \{1, 2, ..., n\}$ and $1 \leq i \leq p$, the sum of each row is

$$\sum_{j=1}^q s_{i,j}^t = \sum_{j=1}^q \left((y-x)\mathrm{sgn}(r_{i,j}^t) + r_{i,j}^t\right)$$
$$= (y-x)\sum_{j=1}^q \mathrm{sgn}(r_{i,j}^t) + \sum_{j=1}^q r_{i,j}^t = 0,$$

and for all $1 \leq j \leq q$, the sum of each column is

$$\sum_{i=1}^p s_{i,j}^t = \sum_{i=1}^p \left((y-x)\mathrm{sgn}(r_{i,j}^t) + r_{i,j}^t\right)$$
$$= (y-x)\sum_{i=1}^p \mathrm{sgn}(r_{i,j}^t) + \sum_{i=1}^p r_{i,j}^t = 0.$$

Thus, S is a centrally n-tuple symmetric (p, q)-rectangle of type y. For any $t \in \{1, 2, ..., n\}$, if $r_{i,j}^t$ is positive, then $r_{i,j}^t = x+m$ for some $m \geq 1$. Hence, $s_{i,j}^t = y+m$ is also positive. Similarly, $r_{i,j}^t$ negative implies $s_{i,j}^t$ negative. Therefore, S is balanced.   □

Proposition 3 If a balanced centrally n-tuple symmetric (p, q)-rectangle exists, then an n-tuple magic (p, q)-rectangle exists.

Proof. Suppose R is the given rectangle. If R has type x, then by Lemma 1, there exists a balanced centrally n-tuple symmetric (p, q)-rectangle of type $-1/2$. Therefore, an n-tuple magic (p, q)-rectangle exists.   □

Example 1 We consider a balanced centrally 5-tuple symmetric $(4, 2)$-rectangle $R := (r_{i,j}^1)(r_{i,j}^2)...(r_{i,j}^5)$ of type 1 as follows.

$$R := \begin{pmatrix} 2 & -2 \\ -3 & 3 \\ -4 & 4 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} 6 & -6 \\ -7 & 7 \\ -8 & 8 \\ 9 & -9 \end{pmatrix}\begin{pmatrix} 10 & -10 \\ -11 & 11 \\ -12 & 12 \\ 13 & -13 \end{pmatrix}\begin{pmatrix} 14 & -14 \\ -15 & 15 \\ -16 & 16 \\ 17 & -17 \end{pmatrix}\begin{pmatrix} 18 & -18 \\ -19 & 19 \\ -20 & 20 \\ 21 & -21 \end{pmatrix}.$$

Then we define a 5-tuple $(4, 2)$-rectangle $S := (s_{i,j}^1)(s_{i,j}^2)$ $...(s_{i,j}^5)$ related to R by

$$s_{i,j}^t = -\frac{3}{2}\mathrm{sgn}(r_{i,j}^t) + r_{i,j}^t, \quad \text{for every} \quad t \in \{1, 2, 3, 4, 5\}.$$

Thus, S is a balanced centrally 5-tuple symmetric $(4, 2)$-rectangle S of type $-1/2$ as follows.

$$S := \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{3}{2} \\ -\frac{5}{2} & \frac{5}{2} \\ \frac{7}{2} & -\frac{7}{2} \end{pmatrix} \begin{pmatrix} \frac{9}{2} & -\frac{9}{2} \\ -\frac{11}{2} & \frac{11}{2} \\ \frac{13}{2} & \frac{13}{2} \\ \frac{15}{2} & -\frac{15}{2} \end{pmatrix} \begin{pmatrix} \frac{17}{2} & -\frac{17}{2} \\ -\frac{19}{2} & \frac{19}{2} \\ \frac{21}{2} & \frac{21}{2} \\ \frac{23}{2} & -\frac{23}{2} \end{pmatrix} \begin{pmatrix} \frac{25}{2} & \frac{25}{2} \\ \frac{27}{2} & \frac{27}{2} \\ \frac{29}{2} & \frac{29}{2} \\ \frac{31}{2} & \frac{31}{2} \end{pmatrix} \begin{pmatrix} \frac{33}{2} & -\frac{33}{2} \\ -\frac{35}{2} & \frac{35}{2} \\ \frac{37}{2} & \frac{37}{2} \\ \frac{39}{2} & \frac{39}{2} \end{pmatrix}.$$

By adding $41/2$ to each entry of S, we obtain a 5-tuple magic $(4, 2)$-rectangle T as below.

$$T := \begin{pmatrix} 21 & 20 \\ 19 & 22 \\ 18 & 23 \\ 24 & 17 \end{pmatrix} \begin{pmatrix} 25 & 16 \\ 15 & 26 \\ 14 & 27 \\ 28 & 13 \end{pmatrix} \begin{pmatrix} 29 & 12 \\ 11 & 30 \\ 10 & 31 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} 33 & 8 \\ 7 & 34 \\ 6 & 35 \\ 36 & 5 \end{pmatrix} \begin{pmatrix} 37 & 4 \\ 3 & 38 \\ 2 & 39 \\ 40 & 1 \end{pmatrix}.$$

Clearly, the sum over each row of any array is 41 and the sum over each column is 82.

Proposition 4 *If a balanced centrally n-tuple symmetric (p_1, q)-rectangle R and a centrally n-tuple symmetric (p_2, q)-rectangle S exist, then a centrally n-tuple symmetric $(p_1 + p_2, q)$-rectangle T exists. If S is a balanced rectangle, then T can also be chosen to be balanced.*

Proof. Suppose S has type x. By Lemma 1, we know that there exists a balanced centrally n-tuple symmetric (p_1, q)-rectangle R' of type $x + np_2q/2$. Then by stacking R' and S together, we obtain a rectangle T whose rows' and columns' sum is zero. Thus, T is a centrally n-tuple symmetric $(p_1 + p_2, q)$-rectangle of type x. If S is balanced, then it is easy to see that T is also balanced. □

Since n-tuple magic (p, q)-rectangles correspond to centrally n-tuple symmetric (p, q)-rectangles of type $-1/2$, we have the following corollary.

Corollary 1 *Suppose an n-tuple magic (p_1, q)-rectangle and a balanced centrally n-tuple symmetric (p_2, q)-rectangle exist. Then an n-tuple magic $(p_1 + p_2, q)$-rectangle exists.*

Using the concept of a centrally n-tuple symmetric rectangle, we can prove the existence of even n-tuple magic rectangles. Our tools are the balanced centrally n-tuple symmetric $(2, 4)$-rectangle $A := (a_{i,j}^1)(a_{i,j}^2)\dots(a_{i,j}^n)$ given by

$$(a_{i,j}^t) = \begin{pmatrix} 4t-3 & -4t+2 & -4t+1 & 4t \\ -4t+3 & 4t-2 & 4t-1 & -4t \end{pmatrix},$$

and the n-tuple magic $(2, 6)$-rectangle $B := (b_{i,j}^1)(b_{i,j}^2)\dots(b_{i,j}^n)$ given by

$$b_{1,j}^t = \begin{cases} 1+12(n-t) & \text{if } j=1, \\ 11+12(n-t) & \text{if } j=2, \\ 3+12(n-t) & \text{if } j=3, \\ 9+12(t-1) & \text{if } j=4, \\ 8+12(t-1) & \text{if } j=5, \\ 7+12(t-1) & \text{if } j=6, \end{cases}$$

and

$$b_{2,j}^t = \begin{cases} 12+12(t-1) & \text{if } j=1, \\ 2+12(t-1) & \text{if } j=2, \\ 10+12(t-1) & \text{if } j=3, \\ 4+12(n-t) & \text{if } j=4, \\ 5+12(n-t) & \text{if } j=5, \\ 6+12(n-t) & \text{if } j=6, \end{cases}$$

for all $t \in \{1, 2, \dots, n\}$.

Proposition 5 *Let $q > 2$ be an even integer. Then an n-tuple magic $(2, q)$-rectangle exists.*

Proof. We induct on q. The existence of n-tuple rectangles A and B shows that we need only prove the proposition for $q \geq 8$. Assume we know that an n-tuple magic $(2, q')$-rectangle exists for all even $q' < q$. Then we know an n-tuple magic $(2, q-4)$-rectangle R exists. By Corollary 1, we can add R and A together to form an n-tuple magic $(2, q)$-rectangle. □

Proposition 6 *Let p and q be even positive integers with $(p, q) \neq (2, 2)$. Then an n-tuple magic (p, q)-rectangle exists.*

Proof. By Proposition 5, we can assume that $q > 2$. Using A and Proposition 4, induction shows that a balanced centrally n-tuple symmetric $(p, 4)$-rectangle R exists. Thus, an n-tuple magic $(p, 4)$-rectangle exists and we can assume that $q > 4$. Now assume that an n-tuple magic (p, q')-rectangle exists for all even $q' < q$. We then know that an n-tuple magic $(p, q-4)$-rectangle S exists. By Corollary 1, we can add R and S together to give an n-tuple magic (p, q)-rectangle. □

Example 2 The following arrays are examples of even n-tuple magic rectangles.

A triple magic $(6, 4)$-rectangle

$$\begin{pmatrix} 1 & 19 & 66 & 60 \\ 50 & 68 & 17 & 11 \\ 3 & 21 & 64 & 58 \\ 70 & 52 & 9 & 15 \\ 23 & 5 & 56 & 62 \\ 72 & 54 & 7 & 13 \end{pmatrix} \begin{pmatrix} 25 & 43 & 42 & 36 \\ 26 & 44 & 41 & 35 \\ 27 & 45 & 40 & 34 \\ 46 & 28 & 33 & 39 \\ 47 & 29 & 32 & 38 \\ 48 & 30 & 31 & 37 \end{pmatrix} \begin{pmatrix} 49 & 67 & 18 & 12 \\ 2 & 20 & 65 & 59 \\ 51 & 69 & 16 & 10 \\ 22 & 4 & 57 & 63 \\ 71 & 53 & 8 & 14 \\ 24 & 6 & 55 & 61 \end{pmatrix}.$$

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Then each row sum of any array is 146 and each column sum of any array is 219.

A 4 -tuple magic (4, 4) -rectangle

$$\begin{pmatrix} 7 & 12 & 49 & 62 \\ 50 & 61 & 8 & 11 \\ 16 & 3 & 58 & 53 \\ 57 & 54 & 15 & 4 \end{pmatrix} \begin{pmatrix} 23 & 28 & 33 & 46 \\ 34 & 45 & 24 & 27 \\ 32 & 19 & 42 & 37 \\ 41 & 38 & 31 & 20 \end{pmatrix} \begin{pmatrix} 39 & 44 & 17 & 30 \\ 18 & 29 & 40 & 43 \\ 48 & 35 & 26 & 21 \\ 25 & 22 & 47 & 36 \end{pmatrix} \begin{pmatrix} 55 & 60 & 1 & 14 \\ 2 & 13 & 56 & 59 \\ 64 & 51 & 10 & 5 \\ 9 & 6 & 63 & 52 \end{pmatrix}.$$

Then each row sum and each column sum of any array in a rectangle equals 130.

III. The *n*-Fold Self-Union of Complete Bipartite Graphs

For any integer $n \geq 1$, the *n -fold self-union* of a graph G, denoted by nG, is the union of n disjoint copies of G. In this section we identify the sufficient condition for degree-magic labellings of the n -fold self-union of complete bipartite graphs $nK_{p,q} = K_{p,q}^1 \cup K_{p,q}^2 \cup ... \cup K_{p,q}^n$ to exist.

Theorem 3 *For any integer $n \geq 1$ and even integers p, $q > 1$, let $K_{p,q}^t$ be the tth copy of $K_{p,q}$ for all $t \in \{1, 2, ..., n\}$. A mapping f from $E(nK_{p,q})$ into positive integers given by*

$$f(u_i^t v_j^t) = r_{i,j}^t \quad for\ every \quad u_i^t v_j^t \in E(K_{p,q}^t),$$

is a d-magic labelling of $nK_{p,q}$ if and only if $R := (r_{i,j}^1)(r_{i,j}^2) ...(r_{i,j}^n)$ is an n -tuple magic (p, q) -rectangle.

Proof. Let $U^t = \{u_1^t, u_2^t, ..., u_p^t\}$ and $V^t = \{v_1^t, v_2^t, ..., v_q^t\}$ be partite sets of $K_{p,q}^t$. Suppose that R is an n -tuple magic (p, q) -rectangle. Then f is a bijection from $E(nK_{p,q})$ onto $\{1, 2, ..., npq\}$. For any $u_i^t \in U^t$, we have

$$f^*(u_i^t) = \sum_{j=1}^{q} f(u_i^t v_j^t) = \sum_{j=1}^{q} r_{i,j}^t$$
$$= \frac{q(npq+1)}{2} = \frac{npq+1}{2} \deg(u_i^t),$$

and for any $v_j^t \in V^t$, we have

$$f^*(v_j^t) = \sum_{i=1}^{p} f(u_i^t v_j^t) = \sum_{i=1}^{p} r_{i,j}^t$$
$$= \frac{p(npq+1)}{2} = \frac{npq+1}{2} \deg(v_j^t).$$

i.e., f is a d-magic labelling of $nK_{p,q}$.

Now suppose that f is a d-magic labelling of $nK_{p,q}$. For all $1 \leq i \neq s \leq p$, we have

$$\sum_{j=1}^{q} r_{i,j}^t = \sum_{j=1}^{q} f(u_i^t v_j^t) = f^*(u_i^t)$$
$$= f^*(u_s^t) = \sum_{j=1}^{q} f(u_s^t v_j^t) = \sum_{j=1}^{q} r_{s,j}^t. \tag{5}$$

For all $1 \leq j \neq z \leq q$, we have

$$\sum_{i=1}^{p} r_{i,j}^t = \sum_{i=1}^{p} f(u_i^t v_j^t) = f^*(v_j^t)$$
$$= f^*(v_z^t) = \sum_{i=1}^{p} f(u_i^t v_z^t) = \sum_{i=1}^{p} r_{i,z}^t. \tag{6}$$

By (5), we have

$$\sum_{j=1}^{q} r_{i,j}^t = \sum_{j=1}^{q} r_{s,j}^t = \frac{q(npq+1)}{2}.$$

By (6), we have

$$\sum_{i=1}^{p} r_{i,j}^t = \sum_{i=1}^{p} r_{i,z}^t = \frac{p(npq+1)}{2}.$$

Therefore, R is an n -tuple magic (p, q) -rectangle. ∎

According to Theorem 3 and Proposition 6, we obtain the following result.

Proposition 7 *Let p and q be even positive integers with $(p, q) \neq (2, 2)$. Then $nK_{p,q}$ is a d-magic graph for all integers $n \geq 1$.*

Example 3 We can construct a d-magic graph $3K_{4,8}$ (see Figure 1) with the labels on edges $u_i^t v_j^t$ of $3K_{4,8}$, where $1 \leq i \leq 4$, $1 \leq j \leq 8$ and $1 \leq t \leq 3$, in TABLE I.

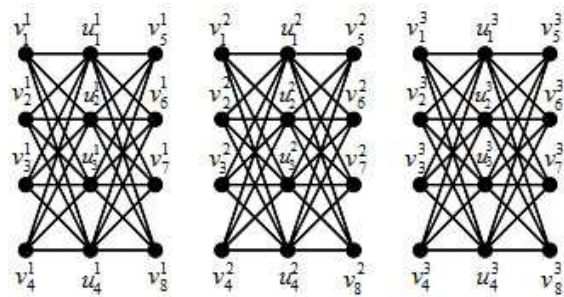Figure 1.   A d-magic graph $3K_{4,8}$.

TABLE I. THE LABELS ON EDGES OF D-MAGIC GRAPH $3K_{4,8}$

Vertices	v_1^1	v_2^1	v_3^1	v_4^1	v_5^1	v_6^1	v_7^1	v_8^1
u_1^1	1	8	93	92	9	16	85	84
u_2^1	6	3	90	95	14	11	82	87
u_3^1	91	94	4	5	83	86	12	13
u_4^1	96	89	7	2	88	81	15	10

Vertices	v_1^2	v_2^2	v_3^2	v_4^2	v_5^2	v_6^2	v_7^2	v_8^2
u_1^2	33	40	61	60	41	48	53	52
u_2^2	38	35	58	63	46	43	50	55
u_3^2	59	62	36	37	51	54	44	45
u_4^2	64	57	39	34	56	49	47	42

Vertices	v_1^3	v_2^3	v_3^3	v_4^3	v_5^3	v_6^3	v_7^3	v_8^3
u_1^3	65	72	29	28	73	80	21	20
u_2^3	70	67	26	31	78	75	18	23
u_3^3	27	30	68	69	19	22	76	77
u_4^3	32	25	71	66	24	17	79	74

Acknowledgments

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