Solving nonlinear Volterra integro-differential equations of fractional order by generalized triangular operational matrix

[Mahnaz Asgari]

Abstract— **In this paper, a computational method based on triangular functions for solving a class of nonlinear Volterra integro-differential equations of fractional order is presented. Generalized operational matrix of triangular functions is constructed and used to transform the equation to a system of algebraic equations. Some numerical examples are provided to illustrate the accuracy and efficiency of the method.**

Keywords— Generalized operational matrix, Fractional integro-differential equation, Triangular functions.

I Introduction

The fractional derivative has been occurring in many physical problems such as, phenomena in electromagnetics, acoustics, the modeling of earthquake, electrochemistry, and material science are described by differential or integro-differential equations of fractional order[1-4]. The reason is a realistic modeling of physical phenomenon, having dependence not only on the time instant, but also on the previous time history, can be successfully achieved by using fractional calculus. Deb et al. in [5,6], proposed orthogonal triangular function (TF) sets derived from the block pulse function (BPF) set. They calculated the operational matrix for integration in TF domain and they established with illustration that the TF domain technique is more accurate than the BPF domain technique as far as integration is concerned.

We used generalized operational matrix of TFs [7] for reducing the nonlinear fractional integro-differential equation to a set of algebraic equations.

Our topic is fractional integro differential equation(FIDE) as following

$$D^\alpha y(t) = f(t) + a(t)y(t) + \int_0^t k(t,s)[y(s)]^n ds, \quad 0 \le t < T,$$

(1)

with these supplementary conditions
$$y^{(i)}(0) = y_i, \quad i = 0,1...,p-1, \quad p-1 < \alpha \le p, \quad p \in N.$$

II. Basic definitions

Some basic definitions and properties of the fractional calculus theory and TF are given in this section.

Definition 2.1 *The Riemann-Liouville fractional integral operator of order* $\alpha \ge 0$ *is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad \alpha > 0, \quad t > 0.$$

$$I^0 f(t) = f(t).$$

Definition 2.2 *The Caputo definition of fractional derivative operator is given by*

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n-1 \le \alpha < n, \ n \in N, \ t > 0.$

Two basic properties are
$$D^\alpha I^\alpha f(t) = f(t),$$

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0^+)\frac{t^k}{k!}, \ t > 0, \ , n-1 \le \alpha < n.$$

Two m-sets of triangular functions are defined over the interval $[0,T)$ as[5]

$$T1_i(t) = \begin{cases} 1 - \dfrac{t-ih}{h} & ih \le t < (i+1)h, \\ 0 & elsewhere. \end{cases}$$

$$T2_i(t) = \begin{cases} \dfrac{t-ih}{h} & ih \le t < (i+1)h, \\ 0 & elsewhere. \end{cases}$$

where, $i = 0,...,m-1$, and $h = \dfrac{T}{m}$.

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The set of TFs may be written as a vector $T1(t)$ and $T2(t)$ of dimension m,

$$T1(t) = [T1_0(t),...,T1_{m-1}(t)]^T,$$

and

$$T2(t) = [T2_0(t),...,T2_{m-1}(t)]^T,$$

then

$$T(t) = [T1(t), T2(t)]^T.$$

The expansion of a function $f(t)$ over $[0,T)$ with respect to TFs , $i = 0,...,m-1$ may be taken as

$$f(t); \quad \hat{f}(t) = F1^T T1(t) + F2^T T2(t) = F^T T(t), \quad (2)$$

where, $F1_i = f(ih)$ and $F2_i = f((i+1)h)$ for $i = 0...m-1$. The vectors $F1, F2$ are called the 1D-TF coefficient vectors and $2m$-vector F is defined as

$$F = [F1, F2]^T.$$

Also, the positive integer powers of a function $f(t)$ may be approximated by 1D-TFs as

$$[f(t)]^n; \quad F_n^T T(t), \quad (3)$$

where F_n is 2m-vector whose elements are nth power of the vector F. It is easily established that[8]

$$T^T(t)BT(t); \quad \hat{B}^T T(t), \quad (4)$$

and

$$T(t)T^T(t)X; \quad \tilde{X}T(t), \quad (5)$$

in which $\hat{B}$ is a 2m-vector with elements equal to the diagonal entries of matrix $\hat{B}$ and $\tilde{X} = diag(X)$ is $2m \times 2m$ diagonal matrix.

Integration of any function $f(t)$ can be approximated as

$$\int_0^t f(s)ds; \quad \int_0^t F^T T(s)ds; \quad F^T PT(t).$$

where, P is operational matrix of integration [6].

III. Generalized operational matrix of TF

In this section, generalized operational matrix of TF for fractional calculus is derived. For $T1_i(t)$, applying definition of the convolution, we get

$$I^\alpha T1_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T1_i(s)ds$$

$$= \frac{1}{\Gamma(\alpha)} \{t^{\alpha-1} * T1_i(t)\}, \ 0 \le t < T,$$

we use Laplace transform to find this integral

$$Ł\{I^\alpha T1_i(t)\} = \frac{1}{\Gamma(\alpha)} Ł\{t^{\alpha-1}\} Ł\{T1_i(t)\},$$

where,

$$Ł\{t^{\alpha-1}\} = \frac{\Gamma(\alpha)}{s^\alpha},$$

and

$$Ł\{T1_i(t)\} = Ł\{u(t-ih) - \frac{t-ih}{h}u(t-ih)$$
$$+ \frac{t-(i+1)h}{h}u(t-(i+1)h)\}$$
$$= [\frac{e^{-ihs}}{s} - \frac{e^{-ihs}}{hs^2} + \frac{e^{-(i+1)hs}}{hs^2}],$$

therefore,

$$Ł\{I^\alpha T1_i(t)\} = \frac{e^{-ihs}}{s^{\alpha+1}} - \frac{e^{-ihs}}{hs^{\alpha+2}} + \frac{e^{-(i+1)hs}}{hs^{\alpha+2}}. \quad (6)$$

Taking inverse Laplace transform of Eq.(6) yields

$$I^\alpha T1_i(t) = \frac{1}{\Gamma(\alpha+2)}((\alpha+1)(t-ih)^\alpha u(t-ih)$$
$$- \frac{(t-ih)^{\alpha+1}}{h}u(t-ih)$$
$$+ \frac{(t-(i+1)h)^{\alpha+1}}{h}u(t-(i+1)h)). \quad (7)$$

In a similar manner, the fractional integration of $T2_i(t)$ is

$$I^\alpha T2_i(t) = \frac{1}{\Gamma(\alpha+2)}((t-ih)^{\alpha+1}u(t-ih)$$
$$- (t-(i+1)h)^{\alpha+1}u(t-(i+1)h)$$
$$- (\alpha+1)(t-(i+1)h)^\alpha u(t-(i+1)h)). \quad (8)$$

Now, we expand $I^\alpha T1_i(t)$ with respect to TFs as follow

$$I^\alpha T1_i(t); \quad [C1_{i0},...,C1_{im-1}]T1(t) + [C2_{i0},...,C2_{im-1}]T2(t),$$

where $C1_{ij} = I^\alpha T1_i(jh)$ and $C2_{ij} = I^\alpha T1_i((j+1)h), \ j = 0,...,m-1.$

From Eq.(7), we get

$$C1_{ij} = 0, \quad j \le i,$$

$$C1_{ij} = \frac{h^\alpha}{\Gamma(\alpha+2)}((\alpha+1)(j-i)^\alpha$$
$$- (j-i)^{\alpha+1} + (j-i-1)^{\alpha+1}), \ i < j,$$

$$C2_{ij} = C1_{ij+1}, \ \forall i, j.$$

Finally, for $i = 0,...,m-1$, $j = 0,...,m-1$, we can write

$$I^{\alpha}T1(t); \quad P1_{\alpha}T1(t) + P2_{\alpha}T2(t), \tag{9}$$

in which

$$P1_{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha+2)} \begin{pmatrix} 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times m},$$

and

$$P2_{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha+2)} \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_1 \end{pmatrix}_{m\times m},$$

where, $\xi_r = (\alpha+1)r^{\alpha} - r^{\alpha+1} + (r-1)^{\alpha+1}$, $r = 0,1,..,m$. Proceeding in a similar manner as for T1, we find that

$$I^{\alpha}T2(t); \quad P3_{\alpha}T1(t) + P4_{\alpha}T2(t), \tag{10}$$

where,

$$P3_{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha+2)} \begin{pmatrix} 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} \\ 0 & 0 & \zeta_1 & \cdots & \zeta_{m-2} \\ 0 & 0 & 0 & \cdots & \zeta_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times m},$$

and

$$P4_{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha+2)} \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \xi_m \\ 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} \\ 0 & 0 & \zeta_1 & \cdots & \zeta_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_1 \end{pmatrix}_{m\times m},$$

where, $\zeta_r = r^{\alpha+1} - (r-1)^{\alpha+1} - (\alpha+1)(r-1)^{\alpha}$. By using Eq.(9) and Eq.(10) fractional integration of $T(t)$ can be computed

$$I^{\alpha}T(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(s)ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \begin{pmatrix} T1(s) \\ T2(s) \end{pmatrix} ds =$$

$$\begin{pmatrix} I^{\alpha}T1(t) \\ I^{\alpha}T2(t) \end{pmatrix}; \begin{pmatrix} P1_{\alpha}T1(t) + P2_{\alpha}T2(t) \\ P3_{\alpha}T1(t) + P4_{\alpha}T2(t) \end{pmatrix}$$

$$= \begin{pmatrix} P1_{\alpha} & P2_{\alpha} \\ P3_{\alpha} & P4_{\alpha} \end{pmatrix} \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix},$$

so,

$$I^{\alpha}T(t) = P_{\alpha}T(t), \tag{11}$$

where P_{α}, generalized operational matrix of T(t), is

$$\begin{pmatrix} P1_{\alpha} & P2_{\alpha} \\ P3_{\alpha} & P4_{\alpha} \end{pmatrix}.$$

The fractional integration of any function $f(t)$ can be approximated as

$$I^{\alpha} f(t); \quad F^T P_{\alpha} T(t).$$

IV Solving nonlinear FIDE

In this section, by using the results of the previous sections, an effective direct method for solving nonlinear FIDE is presented. Using the definitions of the fractional derivatives and integrals, it is suitable to rewrite Eq.(1) in the following form

$$y(t) = f_0(t) + I^{\alpha}(f(t)) + I^{\alpha}(a(t)y(t)) +$$
$$I^{\alpha}(\int_0^t k(t,s)[y(s)]^n ds). \tag{13}$$

where $f_0(t) = \sum_{i=0}^{p-1} y^{(i)}(0^+) \frac{t^i}{i!}$. Approximations of $y(t), f_0(t), f(t), a(t), k(t,s)$, with respect to TFs may be written as

$$f_0(t); \quad F_0^T T(t) = T^T(t)F_0, \tag{14}$$

$$f(t); \quad F^T T(t) = T^T(t)F, \tag{15}$$

$$a(t); \quad A^T T(t) = T^T(t)A, \tag{16}$$

$$y(t); \quad Y^T T(t) = T^T(t)Y, \tag{17}$$

$$[y(t)]^n; \quad Y_n^T T(t) = T^T(t)Y_n, \tag{18}$$

$$k(t,s); \quad T^T(t)KT(s), \tag{19}$$

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where 2m-vectors Y_0, Y, , F , A, Y_n and $2m \times 2m$ matrix K are TF coefficients. Also elements of Y_n are n th power of the vector Y.

From Eqs.(18) and (19), Integral part in Eq.(13) can be approximated as

$$I^\alpha(\int_0^t k(t,s)[y(s)]^n ds); \quad I^\alpha(T^T(t)K\int_0^t T(s)T^T(s)Y_n ds)$$

$$= I^\alpha(T^T(t)K\int_0^t \widetilde{Y}T(s)ds)I^\alpha(T^T(t)K\widetilde{Y}PT(t))$$

$$= I^\alpha(\hat{B}^T T(t)); \quad \hat{B}^T P_\alpha T(t) \tag{20}$$

also

$$I^\alpha(a(t)y(t)); \quad I^\alpha(T^T(t)AY^T T(t)) = I^\alpha(\hat{A}^T T(t))$$

$$= \hat{A}^T P_\alpha T(t). \tag{21}$$

where $\widetilde{Y} = diag(Y_n)$ and $\hat{B}, \hat{A}$ are defined in (7). By substituting Eqs.(14),(17),(20) and (21) in Eq.(13), we get

$$Y^T T(t) = F_0^T T(t) + F^T P_\alpha T(t) + \hat{A}^T P_\alpha T(t) + \hat{B}^T P_\alpha T(t), \tag{22}$$

or,

$$Y - P_\alpha^T F - P_\alpha^T \hat{A} - P_\alpha^T \hat{B} = F_0. \tag{23}$$

Eq.(23) is a nonlinear system of algebraic equations. Components of unknown vector Y can be obtained by solving this system using an iterative method. The presented method is applied on some examples. These examples are selected from different references, so the numerical results can be compared with the other methods.

IV. Numerical Examples

Example 1 *The fractional integro-differential equation is considered as follows*

$$D^{\frac{6}{5}}y(t) = \frac{5}{2\Gamma(0.8)}t^{0.8} - \frac{t^9}{252} + \int_0^t (t-s)^2(y(s))^3 ds, \tag{24}$$

with these supplementary conditions $y(0) = y'(0) = 0.$ *The exact solution is*

$$y(t) = t^2,$$

Table I shows that the numerical solutions derived by presented method are in a very good accuracy with the exact solutions.

Example 2 *Consider the fractional integro-differential equation[9]*

$$D^\alpha y(t) = sinh(t) + \frac{1}{2}cosh(t)sinh(t) - \frac{t}{2} - \int_0^t [y(s)]^2 ds,$$

$$t \in [0,1), 1 < \alpha \le 2, \tag{25}$$

subject to initial conditions $y(0) = 0, y'(0) = 1.$

The value of $\alpha = 2$ is the only case for which we know the exact solution and our approximate solution is in good agreement with the exact values. The exact solution for $\alpha = 2$ is $y(t) = sinh(t)$.

Table I
The absolute error between the exact solution and the approximate solution

m	*Example*1	*Example*2
$m = 8$	2.9×10^{-3}	2.3×10^{-4}
$m = 16$	9.7×10^{-4}	3.1×10^{-5}
$m = 32$	2.3×10^{-4}	3.5×10^{-5}

VI Conclusion

In this paper, we present a computational method based on TFs for solving a class of nonlinear Volterra integrro-differential equations of fractional order. Generalized operational matrix of TF is derived and used to reduce FIDE to a system of algebraic equations. The advantage of this method is low cost of setting up the equations without applying any projection method.

Acknowledgment

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