The new method of research of the systems with increased potential with robust stability

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Abstract — **In this paper we propose the problem of design control systems with increased potential of robust stability in a class of two- parametric structurally stable mappings for objects with m input and n output signals.**

For research of the stability of steady states we using the ideas of the second method A.M. Lyapunov and define components of the gradient vector of the Lyapunov function. During the studying we will represent Equations of state in deviations relative to the stationary states. The general problem of Lyapunov functions for investigation of the systems with improved capacity of robust stability is defined and a condition of stability is given.

We propose a method for construct the control system, built in the two-parameter class of structurally stable, which will be sustained indefinitely in a wide range of uncertain parameters of the control object. This work presents novelty theoretical fundamental results assisting in analyzing of the behavior of control systems, meaning of robust stability.

Keywords — **control system, robust stability, structurally stable mappings, stability of steady States, geometric interpretation, stationary condition.**

I.   Introduction

Relevance of building control systems with a high potential for robust stability for today due to the needs of modern science and technology. In practical problems associated with the design and simulation of control processes in engineering, economics, biology, and other areas in a substantial increase in the capacity of parametric uncertainties robust stability is one of the key factors. This factor ensures the stability of the control system state when released into the chaotic traffic and ensures the applicability of the models and the reliability of the designed control systems. These limits are determined by the region of stability for uncertain system parameters.

In this connection there was necessary to develop models and methods of the control system with an infinitely expandable area of stability in the presence of external and internal disturbances, called control systems with a high potential for robust stability.

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Currently the subject of many studies on the stability of control systems with greater capacity to robust stability with a choice of control laws for dynamic objects in a class of structurally stable maps of catastrophe theory [1,2]. And in these works are mainly limited to obtaining conditions for maintaining the stability properties of a wide range of change the subject and the controller.

The high research interest of robust stability analysis techniques was developed before. Nevertheless, many of them specialized for concrete system of uncertainty structure.

The considered problem is robust controllability of linear system with parametric or non-parametric uncertainties [1,2]. The most important idea in the study of robust stability is to specify constrains for changes in control system parameters. Today we see many works in theory of robust control, which began in the late 1970s and early 1980s and soon developed a number of techniques for dealing with bounded system uncertainty[3-9].

In theory of research of robust stability perform an important function in the theory of control of dynamic objects is [10,11]. The main point of robust stability study is to specify constraints on the change control system parameters that preserve stability. These limits are determined by the region of stability in an uncertain and are selected, i.e. changing parameters [12,13,14,15].

Research of the robust stability of the system is based on the idea of a direct methods A.M. Lyapunov The region of stability is obtained in the form of simple inequalities for uncertain parameters control object and selected regulators properties. In papers [12,13,14] presented a new theoretical methods of robust stability is proposed for linear system with uncertain valued parameters. This method is an extension of the notion of stability where the Lyapunov function is replaced by a geometric interpretation of the Lyapunov function with dependence on the uncertain parameters .

This paper provides a method aimed at usage of universal approach in robust stability analysis for system with parametric uncertainty. The present investigation method is based mainly on the combination of geometrical interpretation of Lyapunov function and the theory of robust stability for control system with increased potential of robust stability for object with *nxn* matrix.

This paper presents the approach of the constructing of Lyapunov function based on the geometrical interpretation of the Lyapunov's direct method (also called the second method of Lyapunov) [9], gradient of dynamical systems in the state space of systems. Also here, expounded a method for constructing Lyapunov's functions to the velocity vector changes in the state of a dynamic control system. If we have

fuzzy degree polynomial form of the Lyapunov function using the Morse lemma in the theory of catastrophes, so the stability of the system is determined by the positive definiteness of a quadratic form.

The concept of building control systems with a high potential for robust stability of dynamic objects is based on the catastrophe theory [3, 4], where the main result are obtained in form of basic structurally stable mappings. They are limited and directly related to the number of control parameters.

The content of this paper is organized in next way: In section 2 we introduce the basic equations of the model at their expanded form. In section 3 we determined stability in sphere of stability of steady state of control system by using geometric interpretation of the Lyapunov's function. In section 4 present definition the conditions of stability control system, built in the class of two-parameter structurally stable mappings. Section 5 contains a concluding remark.

II. **Main mathematical model**

Let's consider the task of building control systems with a high potential for robust stability of a class of two-parametric structurally stable mappings for objects with m inputs and n outputs.

Let the control system to be set in following form

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

Here in $x(t) \in R^n$ – is a state vector of the object, A, B – relatively is matrixes of an object with indefinite options of control of species

$$A = \begin{Vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{Vmatrix},$$

$$B = \begin{Vmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & 0 & \cdots & 0 \\ 0 & 0 & b_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_{nn} \end{Vmatrix}.$$

Control principle is described by the formula in the form of two-parametrical of structurally stable mappings

$$U_i(t) = -x_i^4(t) + k_i' x_i^2(t) + k_i x_i(t), \quad i = 1,2,\dots,n. \tag{2}$$

System (1) in the expanded form is written as

$$\begin{cases} \dot{x}_1(t) = -b_{11}x_1^4(t) + b_{11}k'_1 x_1^2(t) + \\ \quad +(a_{11} + b_{11}k_1)x_1(t) + a_{12}x_2(t) + \\ \quad\quad +a_{13}x_3(t) + \cdots + a_{1n}x_n(t) \\ \dot{x}_2(t) = a_{21}x_1(t) - b_{22}x_2^4(t) + \\ +b_{22}k'_2 x_2^2(t) + (a_{22} + b_{22}k_2)x_2(t) + \\ \quad\quad +a_{23}x_3(t) + \cdots + a_{2n}x_n(t) \\ \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots \\ \dot{x}_n = a_{n1}x_1(t) + a_{n2}x_2(t) + \\ \quad +a_{n3}x_3(t) + \cdots - b_{nn}x_n^4(t) + \\ +b_{nn}k'_n x_n^2(t) + (a_{nn} + b_{nn}k_n)x_n(t) \end{cases} \tag{3}$$

A stationary state of the system (3) is determined by the decision of the equations

$$\begin{cases} -b_{11}x_1^4 + b_{11}k'_1 x_1^2 + (a_{11} + b_{11}k_1)x_1 + a_{12}x_2 + \\ \quad\quad +a_{13}x_3 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 - b_{22}x_2^4 + b_{22}k'_2 x_2^2 + (a_{22} + b_{22}k_2)x_2 + \\ \quad\quad +a_{23}x_3 + \cdots + a_{2n}x_n = 0 \\ \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots - b_{nn}x_n^4 + \\ \quad +b_{nn}k'_n x_n^2 + (a_{nn} + b_{nn}k_n)x_n = 0 \end{cases} \tag{4}$$

From (4) we obtain the stationary States

$$x_{1s}^1 = 0, \ x_{2s}^1 = 0, \ x_{3s}^1 = 0, \dots, x_{ns}^1 = 0, \tag{5}$$

other stationary States will be determined by the solutions of the equations

$$\begin{aligned} &-b_{ii}x_{is}^3 + b_{ii}k'_i x_{is} + (a_{ii} + b_{ii}k_i) = 0, \\ &x_{js} = 0, i \neq j, i = 1,2,\dots,n, \ j = 1,2,\dots,n. \end{aligned} \tag{6}$$

To find solutions of equation (6) we have the principle of scrap. As it is known from the theory of catastrophes [3,4], the solution of the equation (6) corresponds to the critical points of cusp catastrophe, defined by the formula

$$\begin{aligned} f(x_{is}, b_{ii}, k'_i, a_{ii}, k_i) &= -b_{ii}x_{is}^4 + b_{ii}k'_i x_{is}^2 + \\ &+(a_{ii} + b_{ii}k_i)x_{is} = 0, i = 1,2,\dots,n. \end{aligned} \tag{7}$$

Critical, double-degenerated critical and triple-degenerated critical points of cusp catastrophe (7) are determined by equating the first, second and third derivatives of (7) to zero.

Condition (7) is exercised in critical points

$$-4b_{ii}x_{is}^3 + 2b_{ii}k'_i x_{is} + a_{ii} + b_{ii}k_i = 0, \ i = 1,2,\dots,n \tag{8}$$

and

$$-12b_{ii}x_{is}^2 + 2b_{ii}k'_i = 0, \ i = 1,2,\dots,n \tag{9}$$

double-degenerated critical points, and the conditions (8) , (9) and

$$12 b_{ii} x_{is} = 0, \quad i = 1, 2, \ldots, n \tag{10}$$

triple-degenerated critical points.

The position in space of parameters $\left(k'_i, \frac{a_{ii}}{b_{ii}} + k_i \right)$ point, which describes the function with three times a degenerate critical point is determined as

$$(10) \Rightarrow x_{is} = 0 \overset{(9)}{\Rightarrow} k'_i = 0 \overset{(8)}{\Rightarrow} \frac{a_{ii}}{b_{ii}} + k_i, \quad i = 1, 2, \ldots, n. \tag{11}$$

The corresponding function $f(x_{is}, 0, 0) = x_{is}^4$ has three times degenerate point at the origin.

The point of the parameter space, which parameterized the function with a double-degenerate critical points, are determined from the equations (9), (8)

$$(9) \Rightarrow k'_i = 6 x_{is}^2 \overset{(8)}{\Rightarrow} \frac{a_{ii}}{b_{ii}} + k_i = -8 k_{is}^3, \quad i = 1, 2, \ldots, n. \tag{12}$$

If we mark the position of the double-degenerated critical point through x_{is}, then the formula (12) gives the values of the parameters k'_i and $\frac{a_{ii}}{b_{ii}} + k_i$ that describes the function with a double-degenerate critical point x_{is}.

Equation (12) determine the parametric representation of the links between k'_i и $\frac{a_{ii}}{b_{ii}} + k_i$. More direct expression for communication between k'_i and $\frac{a_{ii}}{b_{ii}} + k_i$ can be received if to exclude x_{is} from (12):

$$\left(\frac{k'_i}{6} \right)^{1/2} = x_{is} = \left(-\frac{a_{ii} + b_{ii} k_i}{8 b_{ii}} \right)^{1/3}, i = 1, 2, \ldots, n$$

or

$$\left(\frac{k'_i}{3} \right)^{1/2} = x_{is} = \left(-\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}} \right)^{1/3}, i = 1, 2, \ldots, n$$

$$\left(\frac{k'_i}{3} \right)^3 = -\left(\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}} \right)^2, i = 1, 2, \ldots, n$$

$$\left(\frac{k'_i}{3} \right)^3 + \left(\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}} \right)^2 = 0, \quad i = 1, 2, \ldots, n. \tag{13}$$

As we know from elementary algebra, equation (6) can have up to three real solutions, type:

$$x_{is}^2 = A_i + B_i, x_{is}^{3,4} = -\frac{A_i + B_i}{2} \pm j \frac{A_i + B_i}{2} \sqrt{3}$$

where

$$A_i = \sqrt[3]{\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}} + Q_i}, B_i = \sqrt[3]{\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}} - Q_i},$$

$$Q_i = \left(\frac{k'_i}{3} \right)^3 + \left(\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}} \right)^2, \quad i = 1, 2, \ldots, n.$$

Thus from (13) equation (6) has a solution:

$$x_{is}^2 = 2 \sqrt[3]{\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}}}, x_{js} = 0,$$

$$i \neq j, \ j = 1, 2, \ldots, n, \ i = 1, 2, \ldots, n, \tag{14}$$

$$x_{is}^{3,4} = -\sqrt[3]{\frac{a_{ii} + b_{ii} k_i}{2 b_{ii}}}, x_{js} = 0,$$

$$i \neq j, \ j = 1, 2, \ldots, n, \ i = 1, 2, \ldots, n. \tag{15}$$

Investigation of the stability of steady States (5), (14) and (15) using the ideas of the second method of A. M. Lyapunov, as a research tool which uses special functions, called Lyapunov and is based on two theorems of A.M. Lyapunov [1,5].

III. The geometric approach of the Lyapunov function

Theorems of Lyapunov have the following geometric interpretation. Assume that there exist positive-definite solutions of $V(x_1, x_2, \ldots, x_n)$ for which $(dV/dt < 0)$ full time derivatives less than zero. Consider some cumulative curve of the condition equations (3), beyond the initial moment of time from anywhere in the neighborhood of the steady state.

If dV/dt is the function negatively defined $(dV/dt < 0)$ each integral curve coming from a sufficiently small neighborhood of the stationary States will continue to cross each of the surfaces $V(x_1(t), x_2(t), \ldots, x_n(t)) = c, c = const$, from the outside to the inside, because the function $V(x_1(t), x_2(t), \ldots, x_n(t)) = c$ should continuously decrease. But in this case, the integral curves should unlimited approach the stationary points, thus unperturbed traffic steady asymptotically.

Let suppose that there exists a Lyapunov function, $V_i(x_1, x_2, \ldots, x_n), \ i = 1, 2, \ldots, n$ anti-gradients $-\frac{\partial V_i(x)}{\partial x_j}, i = \overline{1, n}, j = 1, \ldots, n$ specifies the velocity vector of the system (3) i.e. it is assumed that the vector gradient of the Lyapunov functions are aimed in the opposite direction to the velocity vector:

$$\frac{dx_i}{dt} = -\left(\frac{\partial V_i(x)}{\partial x_1} + \frac{\partial V_i(x)}{\partial x_2} + \cdots + \frac{\partial V_i(x)}{\partial x_n} \right).$$

Let's denote the components of the gradient vector for the components of the vector-functions

$$V_1(x_1, x_2, \ldots, x_n), V_2(x_1, x_2, \ldots, x_n), \ldots, V_n(x_1, x_2, \ldots, x_n)$$

thru:

$$\frac{\partial V_1(x)}{\partial x_1} = b_{11} x_1^4 - b_{11} k'_1 x_1^2 - (a_{11} + b_{11} k_1) x_1,$$

$$\frac{\partial V_1(x)}{\partial x_2} = -a_{12} x_2,$$

$$\frac{\partial V_1(x)}{\partial x_3} = -a_{13}x_3, \dots, \frac{\partial V_1(x)}{\partial x_n} = -a_{1n}x_n,$$

$$\frac{\partial V_2(x)}{\partial x_1} = -a_{21}x_3,$$

$$\frac{\partial V_2(x)}{\partial x_2} = b_{22}x_2^4 - b_{22}k'_2x_2^2 - (a_{22} + b_{22}k_2)x_2;$$

$$\frac{\partial V_2(x)}{\partial x_3} = -a_{23}x_3, \dots, \frac{\partial V_2(x)}{\partial x_n} = -a_{2n}x_n$$

$$\begin{matrix} \dots & \dots & \dots & \dots & \dots & \dots \end{matrix}$$

$$\frac{\partial V_n(x)}{\partial x_1} = -a_{n1}x_1, \frac{\partial V_n(x)}{\partial x_2} = -a_{n2}x_2,$$

$$\frac{\partial V_n(x)}{\partial x_3} = -a_{n3}x_3, \dots, \frac{\partial V_n(x)}{\partial x_n} =$$
$$= b_{nn}x_n^4 - b_{nn}k'_nx_n^2 - (a_{nn} + b_{nn}k_n)x_n.$$

Full derivative of the vector Lyapunov function $V(x)$ subject to state equations (3) is defined as the scalar product of the gradient of the Lyapunov function of the velocity vector i.e.

$$\frac{dV(x)}{dt} = -\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial V_i(x)}{\partial x_j}\right)\frac{dx_i}{dt} =$$
$$= -\sum_{i=1}^{n}\begin{bmatrix} a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots - \\ -b_{ii}x_i^4 + b_{ii}k'_ix_i^2 + \\ +(a_{ii}+b_{ii}k_i)x_i + \cdots + \\ +a_{i,n-1}x_{n-1} + a_{in}x_n \end{bmatrix}^2. \quad (16)$$

It follows that the full derivative of the Lyapunov function will always sign-negative function, i.e. sufficient condition for the stability of the stationary States (5) will be executed.

On components of a vector of a gradient we will construct Lyapunov's function, we will look for among functions which as their derivatives on time owing to the set equations of conditions of system (3) represent the sum of functions, each of which depends only on one phase variable.

We obtain the components of vector Lyapunov functions in the form of

$$V_i(x) = -\frac{1}{2}a_{i1}x_1^2 - \frac{1}{2}a_{i2}x_2^2 - \frac{1}{2}a_{i3}x_3^2 -$$
$$-\cdots - \frac{1}{2}a_{i,i-1}x_{i-1}^2 + \frac{1}{5}b_{ii}x_i^5 - \frac{1}{3}b_{ii}k'_ix_i^3 -$$
$$-\frac{1}{2}(a_{ii}+b_{ii}k_i)x_i^2 - \frac{1}{2}a_{i,i+1}x_{i+1}^2 - \cdots - \frac{1}{2}a_{in}x_n^2. \quad (17)$$

Potential function can imagine in the form of

$$V(x) = \sum_{i=1}^{n}\begin{pmatrix} -\frac{1}{2}a_{i1}x_1^2 - \frac{1}{2}a_{i2}x_2^2 - \frac{1}{2}a_{i3}x_3^2 - \\ -\cdots + \frac{1}{5}b_{ii}x_i^5 - \frac{1}{3}b_{ii}k'_ix_i^3 - \\ -\frac{1}{2}(a_{ii}+b_{ii}k_i)x_i^2 - \cdots - \frac{1}{2}a_{in}x_n^2 \end{pmatrix}. \quad (18)$$

The terms of the positive definiteness of a function $V(x)$ from (18) obviously, or we can apply of lemm of Morse [8,9].

The considered system (3) let is in equilibrium (sustainable or unsustainable) i.e. in the steady state, where the gradient of the potential function (the Lyapunov function) $\nabla V = 0$, then the steady states of the system is fair theorem in $\det V_{ij} \neq 0$ and guarantees the existence of a smooth change of variables, such that the Lyapunov function (11) can locally be represented by a quadratic form $V(x) = \sum_{i=N}^{n}\mu_i y_i^2$. It follows that you must calculate the matrix sustainability for comparison equilibrium (5), (14) and (15).

For the presentation of Lyapunov function (18) in the form of quadratic forms with coefficients $\mu_i(i = 1, 2, \dots, n)$, where the positive Lyapunov function will be determined by the signs $\mu_i(\mu_i > 0, i = 1, 2, \dots, n)$ eigenvalue of the matrix sustainability.

$$V_{ij}(x_s) = \left\|\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\bigg|_{x_s}\right\|.$$

It follows that the matrix of stability of the equilibrium points (5), (14) and (15). For this we determine the components of the gradient of vector Lyapunov functions:

$$\frac{\partial V_i(x)}{\partial x_j} = 0 \text{ when } i \neq j, i = 1, 2, \dots, n, \ j = 1, 2, \dots, n$$

$$\frac{\partial V_i(x)}{\partial x_j} = -a_{i1}x_1 - a_{i2}x_2 - a_{i3}x_3 - \cdots +$$
$$+b_{ii}x_i^4 - b_{ii}k'_ix_i^2 - (a_{ii} + b_{ii}k_i)x_i -$$
$$-a_{i,i+1}x_{i+1} - \cdots - a_{in}x_n$$

when $i = j, i = 1, 2, \dots, n, \ j = 1, 2, \dots, n$.

Matrix sustainability (Hesse - Matrix) for the stationary condition (5), we obtain in the form of

$$V_{ij} = \left\|\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\bigg|_{x_s=0}\right\| = \begin{Vmatrix} \mu_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mu_n \end{Vmatrix} \quad (19)$$

where

$$\mu_1 = -(a_{11} + b_{11}k_1 + a_{21} + a_{31} + \cdots + a_{n1}),$$
$$\mu_2 = -(a_{12} + a_{22} + b_{22}k_2 + a_{32} + \cdots + a_{n2}),$$
$$\mu_3 = -(a_{13} + a_{23} + a_{33} + b_{33}k_3 + \cdots + a_{n3}),$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$\mu_n = -(a_{1n} + a_{2n} + a_{3n} + \cdots + a_{nn} + b_{nn}k_n).$$

From this follows that the Liapunov function (11) in the vicinity of a steady state (5) can be represented as a quadratic form

$$-(a_{11} + b_{11}k_1 + a_{21} + a_{31} + \cdots + a_{n1})x_1^2 -$$
$$-(a_{12} + a_{22} + b_{22}k_2 + a_{32} + \cdots + a_{n2})x_2^2 -$$

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$$-(a_{13} + a_{23} + a_{33} + b_{33}k_3 + \cdots + a_{n3})x_3^2 - \cdots -$$
$$-(a_{1n} + a_{2n} + a_{3n} + \cdots + a_{nn} + b_{nn}k_n)x_n^2. \qquad (20)$$

The terms of the positive definiteness of the quadratic form (20), i.e. the stability of the matrix (19) determined by the inequalities

$$
\begin{cases}
a_{11} + b_{11}k_1 + a_{21} + a_{31} + \cdots + a_{n1} < 0, \\
a_{12} + a_{22} + b_{22}k_2 + a_{32} + \cdots + a_{n2} < 0, \\
a_{13} + a_{23} + a_{33} + b_{33}k_3 + \cdots + a_{n3} < 0, \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
a_{1n} + a_{2n} + a_{3n} + \cdots + a_{nn} + b_{nn}k_n < 0.
\end{cases}
\qquad (21)
$$

Thus, the sphere of stability of steady state (5) is determined by the system of inequalities (21).

IV. The robust stability conditions for objects with *nxn* matrix

Let's examine the stability of stationary conditions (14) and (15) based on the method of Lyapunov functions. The equation of state (3) assume deviations relative to the stationary condition (14) and (15). To make this, here is some formalism which allows for representation of equation (3) in deviations relative to the stationary States of (14) and (15).

Let the system state is determined by a set of variables in a column vector X with components $X_1, X_2, \ldots, X_n$. Use this designation, changing system state depending on X can be described in general form $\frac{dX}{dt} = F(x)$.

Here is - a vector, function components, which are equal to the right side of the equation of state (3). Found stationary States of (14), (15) are the solution to the equation
$$F(X_s) = 0.$$

The role of perturbations can be considered, assuming $X = X_s + X$.

Here X is the outrage, i.e. the ratio of the stationary state. With this view of (3) we get the equation for the perturbation X i.e., in the study of stability of any state, the equation of state is written concerning disturbances (deviations from the stationary state) X:
$$\frac{dx}{dt} = F(X_s + X) - F(X_s).$$

This equation must be expanded in a Taylor series, i.e. the right part near the steady state (14) or (15). In our case F has the form of a polynomial in X, then it's always possible. This leads to a finite number of States.

Formally described decomposition can be represented as

$$F(X_s + X) = F(X_s) + \left(\frac{\partial F}{\partial X}\right)_{X_s} \cdot x + \frac{1}{2}\left(\frac{\partial^2 F}{\partial X \partial X}\right)_{X_s} \cdot xx +$$
$$+ \frac{1}{6}\left(\frac{\partial^3 F}{\partial x \partial x \partial x}\right)_{X_s} \cdot xxx + \frac{1}{24}\left(\frac{\partial^4 F}{\partial x \partial x \partial x \partial x}\right)_{X_s} \cdot xxxx + \cdots \quad (22)$$

We determine the value of derivatives in the stationary points X_s

$$\left(\frac{\partial F_i}{\partial X_1}\right)_{X_s} = a_{i1}, \qquad \left(\frac{\partial F_i}{\partial X_2}\right)_{X_s} = a_{i2},$$
$$\left(\frac{\partial F_i}{\partial X_3}\right)_{X_s} = a_{i3}, \ldots \left(\frac{\partial F_i}{\partial X_i}\right)_{X_s} =$$
$$= [-4b_{ii}X_i^3 + 2b_{ii}k'_i x_i + (a_{ii} + b_{ii}k_i)]_{X_s} =$$
$$= \left| k'_i = \sqrt[3]{-\left(\frac{a_{ii} + b_{ii}k_i}{2b_{ii}}\right)^2} \right| =$$
$$= -16(a_{ii} + b_{ii}k_i) - 6(a_{ii} + b_{ii}k_i) + (a_{ii} + b_{ii}k_i) =$$
$$= -21(a_{ii} + b_{ii}k_i), \ldots, \left(\frac{\partial F_i}{\partial X_n}\right)_{X_s} = a_{in}$$

$$\left(\frac{\partial^2 F_i}{\partial X_i \partial X_j}\right)_{X_s} = 0, i \neq j, \qquad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,n$$

$$\left(\frac{\partial^2 F_i}{\partial X_i^2}\right)_{X_s} = [-12b_{ii}X_i^2 + 2b_{ii}k'_i]_{X_s} = -30b_{ii}\sqrt[3]{\left(\frac{a_{ii} + b_{ii}k_i}{2b_{ii}}\right)^2}$$

$$\left(\frac{\partial^3 F_i}{\partial X_i^3}\right)_{X_s} = -24b_{ii}X_i|_{X_s} = -48b_{ii}\sqrt[3]{\frac{a_{ii} + b_{ii}k_i}{2b_{ii}}},$$

$$\left(\frac{\partial^3 F_i}{\partial X_i \partial X_j \partial X_k}\right)_{X_s} = 0, i \neq j \neq k,$$
$$i = 1,2,\ldots,n, \quad j = 1,2,\ldots,n, \quad k = 1,2,\ldots,n$$
$$\left(\frac{\partial^4 F_i}{\partial X_i^4}\right)_{X_s} = -24b_{ii}, \qquad \left(\frac{\partial^4 F_i}{\partial X_i \partial X_j \partial X_k \partial X_l}\right)_{X_s} = 0,$$
$$i \neq j \neq k \neq l, \quad l = 1,2,\ldots,n,$$
$$i = 1,2,\ldots,n, \quad j = 1,2,\ldots,n, \quad k = 1,2,\ldots,n.$$

The equation of state (3) conditions relative to the stationary condition (14) is written

$$\begin{cases} \dot{x}_1(t) = -b_{11}x_1^4(t) - 8b_{11}\sqrt[3]{\frac{a_{11}+b_{11}k_i}{2b_{11}}}x_1^3(t) - \\ \\ -15b_{11}\sqrt[3]{\left(\frac{a_{11}+b_{11}k_i}{2b_{11}}\right)^2}x_1^2(t) - 21(a_{11}+b_{11}k_1)x_1(t) + \\ \\ +a_{12}x_2(t) + a_{13}x_3(t) + \cdots + a_{1n}x_n(t) \\ \\ \dot{x}_2(t) = a_{21}x_1(t) - b_{22}x_2^4(t) - 8b_{22}\sqrt[3]{\frac{a_{22}+b_{22}k_2}{2b_{22}}}x_2^3(t) - \\ \\ -15b_{22}\sqrt[3]{\left(\frac{a_{11}+b_{22}k_2}{2b_{22}}\right)^2}x_2(t) - 21(a_{22}+b_{22}k_2)x_2(t) + \\ \\ +a_{23}x_3(t) + \cdots + a_{2n}x_n(t) \\ \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + a_{n3}x_3(t) + \ldots - \\ \\ -b_{nn}x_n^4(t) - 8b_{nn}\sqrt[3]{\frac{a_{nn}+b_{nn}k_n}{2b_{nn}}}x_n^3(t) - \\ \\ -15b_{nn}\sqrt[3]{\left(\frac{a_{nn}+b_{nn}k_n}{2b_{nn}}\right)^2}x_n^2(t) - 21(a_{nn}+b_{nn}k_n)x_n(t) \end{cases}$$

or

$$\dot{x}_i(t) = a_{i1}x_1(t) + a_{i2}x_2(t) + \cdots - b_{ii}x_i^4(t) - $$
$$8b_{ii}\sqrt[3]{\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}}x_i^3(t) - 15b_{ii}\sqrt[3]{\left(\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}\right)^2}x_i^2(t) - $$
$$- 21(a_{ii}+b_{ii}k_i)x_i(t) + a_{i,i+1}x_{i+1}(t) + \cdots + a_{in}x_n(t). \quad (23)$$

Full derivative of the vector Lyapunov function $V(x)$ taking into account the equations of state in deviations relative to the stationary condition (14) is defined as:

$$\frac{dV(x)}{dt} = -\sum_{i=1}^{n}\begin{bmatrix} a_{i1}x_1 + a_{i2}x_2 + \cdots - b_{ii}x_i^4 - \\ -8b_{ii}\sqrt[3]{\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}}x_i^3(t) - \\ -15b_{ii}\sqrt[3]{\left(\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}\right)^2}x_i^2(t) - \\ -21(a_{ii}+b_{ii}k_i)x_i(t) + \\ +a_{i,i+1}x_i(t) + + \cdots + a_{in}x_n(t) \end{bmatrix}^2 \quad (24)$$

Function (24) is always the sign-negative function. Denote the components of the gradient $\nabla V_i(x)$ from the vector Lyapunov functions:

$$\frac{\partial V_i(x)}{\partial x_1} = -a_{i1}x_1, \frac{\partial V_i(x)}{\partial x_2} = -a_{i2}x_2, \ldots,$$

$$\frac{\partial V_i(t)}{\partial x_i} = b_{ii}x_i^4 + 8b_{ii}\sqrt[3]{\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}}x_i^3 +$$

$$+15b_{ii}\sqrt[3]{\left(\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}\right)^2}x_i^2 + 21(a_{ii}+b_{ii}k_i)x_i,$$

$$\frac{\partial V_i(x)}{\partial x_{i+1}} = -a_{i,i+1}x_{i+1}, \ldots, \frac{\partial V_i(x)}{\partial x_n} = -a_{in}x_n.$$

Hence Lyapunov function we obtain in the form of

$$V(x) = \sum_{i=1}^{n}\begin{pmatrix} -\frac{1}{2}a_{i1}x_1^2 - \frac{1}{2}a_{i2}x_2^2 - \cdots + \frac{1}{5}b_{ii}x_i^5 + \\ \\ +2b_{ii}\sqrt[3]{\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}}x_i^4 + \\ \\ +5b_{ii}\sqrt[3]{\left(\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}\right)^2}x_i^3 + \\ \\ +\frac{21}{2}(a_{ii}+b_{ii}k_i)x_i^2 - \\ \\ -\frac{1}{2}a_{i,i+1}x_{i+1}^2 - \cdots - \frac{1}{2}a_{in}x_n^2 \end{pmatrix} =$$

$$= \sum_{i=1}^{n}\left\{ \begin{array}{l} \frac{1}{5}b_{ii}x_i^5 + 2b_{ii}\sqrt[3]{\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}}x_i^4 + \\ \\ +5b_{ii}\sqrt[3]{\left(\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}\right)^2}x_i^3 - \\ \\ -\frac{1}{2}\begin{bmatrix} a_{i1}x_1^2 + a_{i2}x_2^2 + \cdots - \\ -21(a_{ii}+b_{ii}k_i)x_i^2 + \\ +a_{i,i+1}x_{i+1}^2 + \cdots + a_{in}x_n^2 \end{bmatrix} \end{array} \right\} =$$

$$= \sum_{i=1}^{n}\left\{ \begin{array}{l} \frac{1}{5}b_{ii}x_i^5 + 2b_{ii}\sqrt[3]{\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}}x_i^4 + \\ \\ +5b_{ii}\sqrt[3]{\left(\frac{a_{ii}+b_{ii}k_i}{2b_{ii}}\right)^2}x_i^3 + \\ \\ +\frac{1}{2}\sum_{i=1}^{n}\begin{pmatrix} -a_{1i} - a_{2i} - \cdots + \\ +21(a_{ii}+b_{ii}k_i) - \\ -a_{i+1,i} - \cdots - a_{ni,i} \end{pmatrix}x_i^2 \end{array} \right\}. \quad (25)$$

Basis on Morse's Lemma potential function (25) around the steady state (14) can be represented as a quadratic form

$$\frac{1}{2}\sum_{i=1}^{n}\begin{pmatrix} (-a_{1i} - a_{2i} - \cdots + 21(a_{ii}+b_{ii}k_i) - \\ -a_{i+1,i} - \cdots - a_{ni})x_i^2 \end{pmatrix}. \quad (26)$$

Hence the positive definiteness of Lyapunov functions will be expressed by the inequality

$$-a_{1i} - a_{2i} - \cdots - a_{i-1,i} + 21(a_{ii}+b_{ii}k_i) -$$
$$-a_{i+1,i} - \cdots - a_{n-1,i} - a_{n,i} > 0,$$
$$i = 1,2,\ldots,n. \quad (27)$$

Examine the stability of the stationary state (15). The equation of state (3) anomalies relative to the stationary

condition (15) determine. To do this, calculate the components of the gradient vector:

$$\left(\frac{\partial F_i}{\partial X_1}\right)_{X_s} = a_{i1}, \qquad \left(\frac{\partial F_i}{\partial X_2}\right)_{X_s} = a_{i2}, \qquad \left(\frac{\partial F_i}{\partial X_3}\right)_{X_s} = a_{i3}, \dots,$$

$$\left(\frac{\partial F_i}{\partial X_i}\right)_{X_s^{3,4}} = 2(a_{ii} + b_{ii}k_i) + 3(a_{ii} + b_{ii}k_i) + (a_i + b_{ii}k_i) =$$

$$= 6(a_{ii} + b_{ii}k_i), i = 1, 2, \dots, n,$$

$$\left(\frac{\partial^2 F_i}{\partial X_i^2}\right)_{X_s^{3,4}} = (-12 b_{ii} x_i^2 + 2k'_i)|_{X_s^{3,4}} =$$

$$= -12 b_{ii} \sqrt[3]{\left(\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}\right)^2} - 6\sqrt[3]{\left(\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}\right)^2} =$$

$$= -18 b_{ii} \sqrt[3]{\left(\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}\right)^2}, i = 1, 2, \dots, n,$$

$$\left(\frac{\partial^2 F_i}{\partial x_i \partial x_j}\right)_{X_s} = 0 \text{ when } i \neq j, i = 1, 2, \dots, n, \ j = 1, 2, \dots, n,$$

$$\left(\frac{\partial^3 F_i}{\partial X_i^3}\right)_{X_s^{3,4}} = -24 b_{ii} X_i |_{X_s^{3,4}} = 24 b_{ii} \sqrt[3]{\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}},$$

$$i = 1, 2, \dots, n,$$

$$\frac{\partial^3 F_i}{\partial x_i \partial x_j \partial x_k} = 0 \text{ if } i \neq j \neq k,$$

$$i = 1, 2, \dots, n, j = 1, 2, \dots, n, k = 1, 2, \dots, n,$$

$$\frac{\partial^4 F_i}{\partial X_i^4} = -24 b_{ii}, \ i = 1, 2, \dots, n,$$

$$\frac{\partial^4 F_i}{\partial x_i \partial x_j \partial x_k \partial x_l} = 0, \text{ if } i \neq j \neq k \neq l.$$

The equation of state (3) in deviations relative to the stationary condition (15) is written:

$$\dot{x}_i(t) = a_{i1} x_1(t) + a_{i2} x_2(t) + a_{i3} x_3(t) + \cdots +$$

$$+ a_{i,i-1} x_{i-1}(t) - b_{ii} x_i^4(t) + 4 b_{ii} \sqrt[3]{\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}} x_i^3(t) -$$

$$-9 b_{ii} \sqrt[3]{\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}} x_i^2(t) + 6(a_{ii} + b_{ii}k_i)x_i(t) +$$

$$+ a_{i,i+1} x_{i+1}(t) + \cdots + a_{in} x_n(t), \ i = 1, 2, \dots, n. \qquad (29)$$

Full derivative of the Lyapunov function subject to state equations (29) will be equal to:

$$\frac{dV(x)}{dt} = -\sum_{i=1}^{n} \begin{bmatrix} a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + \cdots + \\ + a_{i,i-1} x_{i-1} - b_{ii} x_i^4 + \\ + 4 b_{ii} \sqrt[3]{\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}} x_i^3 - \\ - 9 b_{ii} \sqrt[3]{\left(\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}\right)^2} x_i^2 + \\ + 6(a_{ii} + b_{ii}k_i)x_i + a_{i,i+1} x_{i+1} \\ + \cdots + a_{i,n-1} x_{n-1} + a_{in} x_n \end{bmatrix}^2, \qquad (30)$$

Function (30) will always be negative sign.
Find components $\nabla V_i(x)$ according to the equation of state (29)

$$\frac{\partial V_i(x)}{\partial X_1} = -a_{i1} x_1, \frac{\partial V_i(x)}{\partial X_2} = -a_{i2} x_2.$$

$$\frac{\partial V_i(x)}{\partial X_3} = -a_{i3} x_3, \dots, \frac{\partial V_i(x)}{\partial X_{i-1}} = -a_{i-1} x_{i-1},$$

$$\frac{\partial V_i(x)}{\partial X_i} = b_{ii} x_i^4 - 4 b_{ii} \sqrt[3]{\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}} x_i^3 +$$

$$+ 9 b_{ii} \sqrt[3]{\left(\frac{a_{ii} + b_{ii}k_i}{2 b_{ii}}\right)^2} x_i^2 - 6(a_{ii} + b_{ii}k_i)x_i,$$

$$\frac{\partial V_i(x)}{\partial X_{i+1}} = -a_{i,i+1} x_{i+1}, \dots, \frac{\partial V_{-1}(x)}{\partial X_{n-1}} = -a_{i,n-1} x_{n-1},$$

$$\frac{\partial V_i(x)}{\partial X_n} = -a_{in} x_n, i = 1, 2, \dots, n.$$

Hence we get the potential function in the form of:

$$V(x) = \sum_{i=1}^{n} \begin{bmatrix} \frac{1}{2} a_{i1} x_1^2 - \frac{1}{2} a_{i2} x_2^2 - \frac{1}{2} a_{i3} x_3^2 - \\ - \cdots - \frac{1}{2} a_{i-1} x_{i-1}^2 + \frac{1}{5} b_{ii} x_i^5 - \\ - b_{ii} \sqrt[3]{\frac{a_i + b_{ii}k_i}{2 b_{ii}}} x_i^4 + \\ + 3 b_{ii} \sqrt[3]{\left(\frac{a_{ii} + b_{ii}k_i}{2 b_i}\right)^2} x_i^3 - \\ - 3(a_{ii} + b_{ii}k_i)x_i^2 - \frac{1}{2} a_{i,i-1} x_{i+1}^2 - \\ - \cdots - \frac{1}{2} a_{i,n-1} x_{n-1}^2 - \frac{1}{2} a_{i,n} x_n^2 \end{bmatrix}, \qquad (31)$$

By Morse's lemma [8,9] potential function (31) using the Hesse matrix and replacement variables lead to a quadratic form:

$$V(x) = \frac{1}{2} \sum_{i=1}^{n} \begin{pmatrix} -a_{1i} - a_{2i} - a_{3i} - \cdots - a_{i-1,i} - \\ -3(a_{ii} + b_{ii}k_i) - a_{i+1,i} - \\ - \cdots - a_{n-1,i} - a_{n,i})x_i^2 \end{pmatrix}.$$

Conditions for stability of Hesse matrix will be expressed as a system of inequalities.

$$-\left(\begin{matrix} a_{1i}+a_{2i}+a_{3i}+\cdots+a_{i-1,i}+ \\ 3(a_{ii}+b_{ii}k_i)+a_{i+1,i}+\cdots+a_{n-1,i}+a_{ni} \end{matrix}\right)>0,$$
$$i=1,2,\ldots,n. \tag{32}$$

Thus, the control system, built in the class of two-parametrical structurally stable mappings will be sustained indefinitely wide range of uncertain parameters control object $a_i(i=1,2,\ldots,n)$. Stationary States (5) exists and stable changes in the unspecified object in the field of (14), and the stationary States of (14) and (15) appears when the loss of stability condition (5) and they simultaneously do not exist. Among the stationary States of (14) and (15), stationary condition (14) is stable at executions of the system of inequalities (27). It should be noted (32) what exists is a condition (5) and the execution of a system of inequalities (27) is unsustainable.

V.   Concluding remarks

The results obtained by designing control system with increased potential of robust stability and dynamics. Allow to assure safety and efficiency of control systems in engineering and technology during their construction and operation.

We also developed and approach to designing control systems with increased potential of robust stability for linear objects with uncertain parameters by using control laws in the two-parameter class of structurally stable mapping.

In result, it will help to increase the robust stability when design automated control systems. Therefore, the quality of control of technological process can be improved, which is especially desired when operating in present of uncertainties data.

The theoretical importance of these results should motivate new fundamental studies on typical application techniques, clarification area of robust control and stability [16,17,18].

In this paper we have considered the new method of robust stability to solve complex control system by using a Lyapunov function with some moments from catastrophe theory. Hence the proposed method is suitable for solving problems of decision making in various areas, like automated systems.

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