The existence of positive solutions for three-point boundary value problems of fractional differential equations

Li Zou, Shenjie Fan, Jiushuan Zhang, Lizheng Xue, Dong Pan, Shurong Sun*

Abstract—**In this paper, we investigate the existence of solutions for a class of three-point boundary value problems of fractional differential equations. By using Lebegesgue dominated convergence theorem and schauder fixed point theorem, a sufficient condition of the existence solution is obtained.**

Keywords—**fixed point theorem; dominated convergence theorem; the existence of solutions**

1. Introduction

Fractional calculus, the study of integration and differentiation of fractional order, has recently been extended to include its discrete analogues of fractional difference calculus and fractional quantum calculus. Due to the similarities of the three theories there has been research on whether there exist a single theory that encapsulates them.

Fractional differential equations are extention and expansion of integer-order differential equations. At present, the fractional differential equations have been increasingly applied to different fields of sciences.

Fractional differential equations are using widely in the fields of various sciences such as physics, mechanic, chemistry, engineering. Fractional differential equations have been of increasing importance for the past decades due to their diverse applications in science and engineering, such as, the memory of a variety of materials, signal identification and image processing, optical systems, thermal system materials and mechanical systems, control system. Many interesting results of the existence of solutions of various classes of fractional differential equations have been obtained, see [1-24] and the references therein.

Li Zou, Shenjie Fan, Jiushuan Zhang, Lizheng Xue, Dong Pan, School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P R China.

*Corresponding author: Shurong Sun, School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P R China.

Many scholars paid more attention to it, and there many interesting results on the existence of solutions of various classes of fractional differential equations have been obtained.

Recently, much attention has been focused on the study of the existence and multiplicity of solutions or positive solutions for boundary value problems of fractional differential equations with boundary value problems by the use of techniques of nonlinear analysis (fixed-point theorems, Leray--Schauder theory, the upper and lower solution method.

Boundary value problems of the fractional differential equations are branches of the fractional differential equations and have received much attention. There have been many results for boundary value problems of the fractional differential equations, such as the solutions for boundary value problem of fractional differential equations and m-point boundary value problems [1].

Gan in [18] investigated the existence and non-existence of positive solution for contains the parameters of two-point boundary value problems of fractional differential equations by using fixed point theorem on compression and expansion of cones. Liang and Zhang in [19] deals with the existence of positive solutions for a class of two-point boundary value problems of fractional differential equations by means of upper and lower solutions method and fixed point theorem. Xu 、Xian and Zhou in [20] investigate the existence of solutions for a class of periodic boundary value problems of fractional differential equations by using contraction mapping principle.

Inspired by the above document, we investigate the following the existence of solutions of fractional differential equations.

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t,u(t)), & t \in J = [0,T], \\ u(0) = 0, u(T) = b\,u(\eta), \end{cases} \quad (1)$$

where $1 < \alpha \le 2$, $0 < b < 1$, $\eta \in (0,T)$, $f : J \times R \times R \to R$, this paper provides a sufficient condition of the existence of solution for the problem (1) by using schauder fixed point theorem.

2. Preliminaries

Definition 2.1. The fractional integral of order α for a function $y : (0,\infty) \to R$ is defined as:

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

where $\alpha > 0$, $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2. [1] For a continous function
$$y : (0,\infty) \to R,$$
the Captuo derivative of fractional order α is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-a-1} y^{(n)}(s)ds,$$

where $\alpha > 0$, $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α, $\Gamma(\cdot)$ denotes the Gamma function.

Lemma 2.1. [1] Let $\alpha > 0$, if $u \in C(0,1) \bigcap L^1(0,1)$, $D_{0+}^{\alpha} u(t) \in C(0,1) \bigcap L^1(0,1)$, for some $c_i \in \Re, i = 1,2,3 \dots N,$ subject to

$$I_{0+}^{\alpha} {}^c D_{0+}^{\alpha} u(t) = u(t) + c_1 + c_2 t + \cdots + c_N t^{n-1},$$
$$c_i \in \Re, i = 1,2,\cdots,n,$$

where $n = [\alpha]+1$.

Lemma 2.2. Let $h \in C(J,R)$, a function u is a solution of the fractional BVP

$$\begin{cases} {}^c D_{0+}^{\alpha} u(t) = h(t), & t \in J, 1 < \alpha \leq 2, \\ u(0) = 0, u(T) = b\,u(\eta). \end{cases} \quad (2)$$

If and only if u is a solution of the fractional integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds$$
$$+ \frac{ct}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds$$
$$- \frac{cbt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s)ds,$$

where $c = \dfrac{1}{\eta - T}$.

Proof: By Lemma 2.1，transform the problem（2）into a integral equation

$$u(t) = I_{0+}^{\alpha} h(t) + c_1 + c_2 t, \qquad c_1, c_2 \in R,$$

by the boundary value conditions
$$u(0) = 0, \ u(T) = b\,u(\eta),$$
we can get

$$\begin{cases} c_1 = 0, \\ c_1 = \dfrac{ct}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds - \dfrac{cbt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s)ds. \end{cases}$$

where $c = \dfrac{1}{\eta - T}$. Thus

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds$$
$$+ \frac{ct}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds$$
$$- \frac{cbt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s)ds.$$

They are equivalent.

Measurable function $m : J \to R,$ defined norm

$$\|m\|_{L^p(J,R)}$$

$$= \begin{cases} (\int |m(t)|^p dt)^{\frac{1}{p}}, & 0 \leq p \prec \infty, \\ \inf_{\mu(\bar{J})} \left\{ \sup_{t \in J - \bar{J}} |m(t)| \right\}, & p = \infty. \end{cases}$$

where $\mu(\bar{J})$ is Lebegesgue measure on J, $L^p(J,R)$ is a Banach space consists of all Lebegesgue measureable function $m : J \to R$, $\|m\|_{L^p(J,R)} < \infty$.

Lemma 2.3. [1] Lebegesgue dominated convergence theorem:

Supposes that $E \subseteq R^q$ is measurable set，$\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable function on E. F is nonnegtive L integrable function on E, if for any natural number n,

$$|f_n(x)| \leq F(x) \, \text{a.e on } E,$$
$$\lim_{n \to \infty} f_n(x) = f(x),$$

thus

$$\lim_{n \to \infty} \int_E |f_n(x) - f(x)| dx = 0,$$
$$\lim_{n \to \infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Lemma 2.4. [1] Schaefer fixed point theorem. Supposes X is a Banach space, mapping $T : X \to X$ is a completely continous mapping. If set

$$V = \left\{ u \in X \big| u = \mu T u, 0 \leq \mu \prec 1 \right\}$$

is a bounded set, then mapping T has a fixed point in X.

3. Main Results

Definition: $X = C(J,R)$, its norm is $\|u\| = \sup_{t \in J} |u(t)|$, it is obvious that X is a Banach space. We provide a sufficient condition of the existence of solution

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for the problem (1) by Schaefer fixed point theorem. We make the following conditions through out this paper:

(H_1) $f : J \times R \times R \to R$ is continuous

(H_2) there exists a constant $L_1 > 0$, for $\forall t \in J$ and for all $u \in X$,

$$\left| f(t, u(t)) \right| \le L_1.$$

Theorem 3.1 Assuming the conditions of the (H_1) and (H_2) are satisfied. Then the BVP (1) has a positive solutions at least.

Proof: By Lemma 2.2, the promble (1) is equivalent to

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds$$
$$+ \frac{ct}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, u(s)) ds$$
$$- \frac{cbt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, u(s)) ds.$$

Considering the mapping $F : X \to X$ is followed.

$$(Fu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds$$
$$+ \frac{ct}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, u(s)) ds$$
$$- \frac{cbt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, u(s)) ds,$$

therefore, the existence of positive solutions for three-point boundary value problems (1) are turning to problems of the existence of fixed point that mapping F in X.

Fristly, proving F is continuous.

Take the series $\{u_n\} \subset X$, let the $u_n \to u\ (n \to \infty)$, by the Lebegesgue dominated convergence theorem, we can get $\forall t \in J$, there is

$$\left| (Fu_n)(t) - (Fu)(t) \right|$$
$$\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f(s, u_n(s)) - f(s, u(s)) \right\| ds$$
$$+ \frac{ct}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left\| f(s, u_n(s)) - f(s, u(s)) \right\| ds$$
$$+ \frac{cbt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \left\| f(s, u_n(s)) - f(s, u(s)) \right\| ds$$
$$\le \frac{T^\alpha + (1+b) c T^{\alpha+1}}{\Gamma(\alpha+1)} \left\| f(s, u_n(s)) - f(s, u(s)) \right\|,$$

by the conditions of the (H_1) : f is continuous, so

$$\left\| f(s, u_n(s)) - f(s, u(s)) \right\| \to 0,$$

therefore we can draw a conclusion that F is continuous.

And then, we prove that F mapped a bounded set to a bounded set in X.

Let $\Omega \subset X$ is a bounded set. $\forall u \in \Omega$, Combining with (H_2), we have

$$\left| (Fu)(t) \right| \le \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$
$$+ \frac{cL_1 t}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds + \frac{cL_1 bt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} ds$$
$$\le \frac{L_1 t^\alpha}{\alpha \Gamma(\alpha)} + \frac{cL_1 T^\alpha}{\alpha \Gamma(\alpha)} + \frac{cbL_1 \eta^\alpha}{\alpha \Gamma(\alpha)}$$
$$\le \frac{L_1 T^\alpha + (1+b) L_1 c T^{\alpha+1}}{\Gamma(\alpha+1)} = L_2.$$

Namely, there is constant $L_2 > 0$, make $\|Fu\| \le L_2$, thus, we know that F mapped a bounded set to a bounded.

Finally, we will prove that the mapping F is equicontinuous.

Set $0 < t_1 < t_2 < T$, for $\forall u \in X$, by (H_2) we have

$$\left| (Fu)(t_2) - (Fu)(t_1) \right|$$
$$\le \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t-s)^{\alpha-1} f(s, u(s)) ds - \int_0^{t_1} (t-s)^{\alpha-1} f(s, u(s)) ds \right|$$
$$+ \frac{c(t_2 - t_1)}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left| f(s, u(s)) \right| ds$$
$$+ \frac{cb(t_2 - t_1)}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \left| f(s, u(s)) \right| ds$$
$$\le \frac{L_1}{\Gamma(\alpha)} (t_2^\alpha - t_1^\alpha) + \frac{c\eta^\alpha L_1 (t_2 - t_1)}{\alpha \Gamma(\alpha)} + \frac{cT^\alpha L_1 b(t_2 - t_1)}{\Gamma(\alpha)}$$
$$\le \frac{L_1}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha) + \frac{c(1+b) T^\alpha L_1 (t_2 - t_1)}{\Gamma(\alpha+1)},$$

when $t_1 \to t_2$, we have

$$\left| (Fu)(t_2) - (Fu)(t_1) \right| \to 0,$$

so F is equicontinuous on J. According to Ascoli-Arzela theorem, F is completely continuous.

Meanwhile, we prove that set

$$V = \left\{ u \in X, u = \mu Fu, 0 < \mu < 1 \right\}$$

is bounded set.

Let $u \in V$, then, $u = \mu Fu, 0 < \mu < 1$, for $\forall t \in J$, we have

$$\left| u(t) \right| = \left| \mu (Fu)(t) \right| \le \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$
$$+ \frac{cL_1 t}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds + \frac{cL_1 bt}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} ds$$
$$\le \frac{L_1 T^\alpha + 2cL_1 T^{\alpha+1}}{\Gamma(\alpha+1)} = L_3.$$

There is a constant $L_3 > 0$, make $\|u\| \le L_3$, then we know that the set V is bounded set.

Therefore, in summary, schauder fixed point theorem shows, F in X has at least one fixed point, the boundary conditions (1) has at least one solution.

This proves that the existence of solutions with the boundary condition (1) .

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