Existence results for fractional *q*-difference equations with nonlocal *q*-integral boundary conditions

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Abstract—**In this paper, we discuss the existence of positive solutions for nonlocal *q*-integral boundary value problems of fractional *q*-difference equations. By applying the generalized Banach contraction principle, the monotone iterative method, and Krasnoselskii's fixed point theorem, some existence results of positive solutions are obtained.**

Keywords—**Positive solutions; boundary value problems; fractional; *q*-difference equations**

1. Introduction

The *q*-difference calculus or quantum calculus is an old subject. Studies on *q*-difference equations appeared already at the beginning of the twentieth century in intensive works especially by Jackson [1], Carmichael [2] and other authors such as Poincare, Picard, Ramanujan.

Up to date, fractional differential equations have evolved into a multidisciplinary subject, for example, see [3–30] and the references therein.

For some recent work on *q*-difference equations and fractional *q*-difference equations, we refer the reader to the papers [7–18, 20, 24, 29, 30], and basic definitions and properties of *q*-difference calculus can be found in the book [19]. The fractional *q*-difference calculus had its origin in the works by Al-Salam [20] and Agarwal [21].

Ahmad, Ntouyas, and Purnaras [10] studied the following nonlinear fractional *q*-difference equation

$$_C D_q^\alpha u(t) = f(t, u(t)), 0 \le t \le 1, 1 < \alpha \le 2,$$

$$a_1 u(0) - b_1 D_q u(0) = c_1 u(\eta_1),$$

$$a_2 u(1) + b_2 D_q u(1) = c_2 u(\eta_2),$$

where $_C D_q^\alpha$ is the fractional *q*-derivative of the Caputo type.

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The existence of solutions for the problem is shown by applying some well-known tools of fixed point theory such as Banach's contraction principle, Krasnoselskii's fixed point theorem, and the Leray-Schauder nonlinear alternative.

In this paper, we deal with the following nonlocal *q*-integral boundary value problem of nonlinear fractional *q*-derivatives equation:

$$_C D_q^\alpha u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \quad u(0) = 0, \quad (1.1)$$

$$u(1) = \mu I_q^\beta u(\eta) = \mu \int_0^\eta \frac{(\eta - qs)^{(\beta-1)}}{\Gamma_q(\beta)} u(s) d_q s,$$

where

$$q \in (0,1), \quad 2 < \alpha < 3, 0 < \beta \le 2, 0 < \eta < 1, \mu > 0, \quad _C D_q^\alpha \text{ is}$$

the *q*-derivative of Caputo type of order α, $f \in C(R^+, R^+)$.

In the present work, we gave the corresponding Green's function of the boundary value problem and its properties. By using the Krasnoselskii's fixed point theorem, some existence results of positive solutions to the above boundary value problems are enunciated.

2. Preliminaries

The following formulas will be used later, namely, the integration by parts formula

$$\int_0^x f(s)(D_q g)(s) d_q s = [f(s) g(s)]_{s=0}^{s=x} - \int_0^x (D_q f)(s) g(qs) d_q s,$$

$$[a(t-s)]^{(\gamma)} = a^\gamma (t-s)^{(\gamma)},$$

$$_t D_q (t-s)^{(\gamma)} = [\gamma]_q (t-s)^{(\gamma-1)},$$

$$_s D_q (t-s)^{(r)} = -[\gamma]_q (t-qs)^{(\gamma-1)},$$

$$\left(_x D_q \int_0^x f(x,s) d_q s \right)(x) = \int_0^x {}_x D_q f(x,s) d_q s + f(qx, x),$$

where $_t D_q$ denotes the derivative with respect to the variable t.

Definition 2.1 Let $\alpha \geq 0$ and f be a function defined on [0,1]. The fractional q-integral of Riemann-Liouville type is

$$\left(I_q^0 f\right)(x) = f(x),$$

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} f(s) d_q s,$$

$\alpha > 0, x \in [0,1],$

Definition 2.2 The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined

$$\left(D_q^0 f\right)(x) = f(x),$$

$$\left(D_q^\alpha f\right)(x) = \left(D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f\right)(x),$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma 2.3 Assume that $\gamma \geq 0$ and $a \leq b \leq t,$ then

$$(t - a)^{(\gamma)} \geq (t - b)^{(\gamma)}.$$

Lemma 2.4 Let $\alpha, \beta \geq 0$ and f be a function defined on [0,1]. Then the following formulas hold

$$\left(I_q^\beta I_q^\alpha f\right)(x) = \left(I_q^{\alpha+\beta} f\right)(x),$$

$$\left(D_q^\alpha I_q^\alpha f\right)(x) = f(x).$$

Lemma 2.5 Let $\alpha > 0$ and n be a positive integer. Then the following equality holds:

$$\left(I_q^\alpha D_q^n f\right)(x) = \left(D_q^n I_q^\alpha f\right)(x)$$

$$- \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} \left(D_q^k f\right)(0).$$

Lemma 2.6 Let $\alpha \in R^+, \quad \lambda \in (-1, +\infty),$ the following is valid:

$$I_q^\alpha \left((t-a)^{(\lambda)}\right) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (t-a)^{(\alpha+\lambda)},$$

$0 < a < t < b.$

Lemma 2.7 Let

$$M = \Gamma_q(3+\beta) - \mu \Gamma_q(3)\eta^{2+\beta} > 0.$$

Then, for a given $h \in C[0,1]$, the unique solution of the boundary value problem

$${}^c D_q^\alpha u(t) + h(t) = 0, t \in (0,1), 2 < \alpha \leq 3,$$

subject to the boundary condition

$$u(0) = 0, u'(0) = 0,$$

$$u(1) = \mu I_q^\beta u(\eta) = \mu \int_0^\eta \frac{(\eta - qs)^{(\beta-1)}}{\Gamma_q(\beta)} u(s) d_q s,$$

$$0 < \beta \leq 2, 0 < \eta < 1,$$

$$u(t) = \int_0^1 G(t, qs) h(s) d_q s,$$

where $G(t, qs) = g(t, qs) + \frac{\mu t^2}{M} H(\eta, qs),$

$$g(t,s) = \begin{cases} \dfrac{t^2(1-s)^{(\alpha-1)} - (t-s)^{\alpha-1}}{\Gamma_q(\alpha)}, 0 < s \leq t < 1, \\[3mm] \dfrac{t^2(1-s)^{(\alpha-1)}}{\Gamma_q(\alpha)}, 0 < t \leq s < 1, \end{cases}$$

$$H(\eta,s) = \begin{cases} \dfrac{\Gamma_q(3)(1-s)^{(\alpha-1)}\eta^{2+\beta}}{\Gamma_q(\alpha)} - \dfrac{\Gamma_q(3+\beta)(\eta-s)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}, 0 < s \leq \eta < 1, \\[3mm] \dfrac{\Gamma_q(3)(1-s)^{(\alpha-1)}\eta^{2+\beta}}{\Gamma_q(\alpha)}, 0 < \eta \leq s < 1. \end{cases}$$

Proof: ${}^c D_q^\alpha u(t) = -h(t) \Leftrightarrow I_q^\alpha {}^c D_q^\alpha u(t) = -I_q^\alpha h(t).$

We have

$$u(t) = -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + c_2 t^2 + c_1 t + c_0$$

$$u(0) = 0, c_0 = 0.\ u'(0) = 0,$$

$$u'(t) = -[\alpha-1]_q \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} h(s) d_q s + 2c_2 t + c_1.$$

We have

$$c_1 = 0,\ u(t) = -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + c_2 t^2$$

$$u(1) = -\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + c_2,$$

subject to the boundary condition

$$u(1) = \mu I_q^\beta u(\eta) = \mu \int_0^\eta \frac{(\eta - qs)^{(\beta-1)}}{\Gamma_q(\beta)} u(s) d_q s.$$

We have

$$I_q^\beta u(\eta) = \int_0^\eta \frac{(\eta-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \left(-\int_0^s \frac{(s-xq)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(x) d_q x + c_2 s\right) d_q s$$

$$= -\int_0^\eta \frac{(\eta-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \left(\int_0^s \frac{(s-xq)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(x) d_q x\right) d_q s$$

$$+ \int_0^\eta \frac{(\eta-qs)^{(\beta-1)}}{\Gamma_q(\beta)} c_2 s d_q s,$$

$$\mu I_q^\beta u(\eta) = -\mu \int_0^\eta \frac{(\eta-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \left(\int_0^s \frac{(s-xq)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(x) d_q x\right) d_q s$$

$$+ \mu \int_0^\eta \frac{(\eta-qs)^{(\beta-1)}}{\Gamma_q(\beta)} c_2 s d_q s$$

$$= \mu \left(-\int_0^\eta \frac{(\eta-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} h(s) d_q s + \frac{\Gamma_q(3)c_2}{\Gamma_q(3+\beta)} \eta^{2+\beta}\right),$$

$$u(1) = \mu I_q^\beta u(\eta).$$

Hence

$$-\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)d_q s + c_2$$

$$= \mu(-\int_0^\eta \frac{(\eta-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} h(s)d_q s + \frac{\Gamma_q(3)c_2}{\Gamma_q(3+\beta)}\eta^{2+\beta}).$$

We have

$$(1-\frac{\mu\Gamma_q(3)\eta^{2+\beta}}{\Gamma_q(3+\beta)})c_2 = \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s$$

$$-\mu\int_0^\eta \frac{(\eta-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}h(s)d_q s.$$

$$c_2 = \frac{\Gamma_q(3+\beta)}{\Gamma_q(3+\beta)-\mu\Gamma_q(3)\eta^{2+\beta}}(\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s$$

$$-\mu\int_0^\eta \frac{(\eta-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}h(s)d_q s).$$

Let

$$M = \Gamma_q(3+\beta)-\mu\Gamma_q(3)\eta^{2+\beta},$$

hence

$$c_2 = \frac{\Gamma_q(3+\beta)}{M}(\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s$$

$$-\mu\int_0^\eta \frac{(\eta-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}h(s)d_q s),$$

$$u(t) = -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s$$

$$+t^2\frac{\Gamma_q(3+\beta)}{M}(\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s$$

$$-\mu\int_0^\eta \frac{(\eta-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}h(s)d_q s)$$

$$= [t^2\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s]$$

$$+[\frac{\mu\Gamma_q(3)\eta^{2+\beta}}{M}t^2\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_q s$$

$$-\frac{\mu\Gamma_q(3+\beta)t^2}{M}\int_0^\eta \frac{(\eta-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha)}h(s)d_q s]$$

$$= \int_0^1 g(t,qs)h(s)d_q s + \frac{\mu t^2}{M}\int_0^1 H(\eta,qs)h(s)d_q s$$

$$= \int_0^1 G(t,qs)h(s)d_q s.$$

Lemma 2.8 The function $g(t,s)$ satisfies the following properties:

$$g(t,qs) \geq 0, g(t,qs) \leq g(qs,qs) \quad 0 \leq t,s \leq 1$$

Lemma 2.9 The function $G(t,qs)$ satisfies the following properties:

(i) G is a continuous function and

$$G(t,qs) \geq 0, \text{ for } (t,s) \in [0,1]\times[0,1].$$

(ii) There exists a positive function $\rho \in C((0,1),(0,+\infty))$, such that $\max\limits_{0\leq t\leq 1} G(t,qs) \leq \rho(s)$, where

$$\rho(s) = g(qs,qs)+\frac{\mu}{M}H(\eta,qs), s \in (0,1).$$

3. Main Results

Let $X = C[0,1]$ be a Banach space endowed with the norm

$$\|u\|_x = \max\limits_{0\leq t\leq 1}|u(t)|.$$

Define $P = \{u \in X, u(t) \geq 0, 0 \leq t \leq 1\}$, Define the operator $T : P \rightarrow X$ as follows::

$$(\mathsf{T}u)(t) = \int_0^1 G(t,qs)f(s,u(s))d_q s. \qquad (3.1)$$

It follows from the nonnegativeness and continuity of G and f that the operator $T : P \rightarrow X$ satisfies $T(P) \subset P$ and is completely continuous.

Theorem 3.1 Suppose that there exists $0<\xi_1 <\xi_2$ such that (H_f) $f(t,u):[0,1]\times[0,\xi_2]\rightarrow R$ is continuous and nondecreasing relative to u, and

$$\max\limits_{0\leq t\leq 1} f(t,\xi_2) \leq L\xi_2, \qquad (3.2)$$

$$\min\limits_{\tau_1\leq t\leq \tau_2} f(t,\xi_1) \geq l\xi_1. \qquad (3.3)$$

Then the BVP (1.1) has one positive solution u^* satisfying

$$\xi_1 \leq \|u^*\|_\chi \leq \xi_2,$$

and

$$\lim\limits_{m\to\infty} T^m u_0 = u^*, u_0(t) = \xi_2.$$

Proof: We denote $\Omega[\xi_1,\xi_2] = \{u \in P : \xi_1 \leq U \leq \xi_2\}$. In what follows, we first show that T :

$$\Omega[\xi_1,\xi_2] \rightarrow \Omega[\xi_1,\xi_2].$$

Let $u \in \Omega[\xi_1,\xi_2]$, then $0<\xi_1\leq u(t)\leq\|u\|_X\leq\xi_2.$

By assumption (H_f), we have

$$0 \leq f(t,u(t)) \leq f(t,\xi_2) \leq \max\limits_{0\leq t\leq 1} f(t,\xi_2) \leq L\xi_2, 0 \leq t \leq 1.$$

$$f(t,u(t)) \geq f(t,\xi_1) \geq \min\limits_{\tau_1\leq t\leq \tau_2} f(t,\xi_1) \geq l\xi_1$$

Hence, for any $u \in \Omega[\xi_1,\xi_2]$,

$$\|Tu\|_\chi = \max\limits_{0\leq t\leq 1}\int_0^1 |G(t,qs)f(s,u(s))|d_q s$$

$$\leq \int_0^1 p(s)f(s.u(s))d_q s \leq L\xi_2\int_0^1 p(s)d_q s,$$

$$L\xi_2\int_0^1(\frac{q^2(1-qs)^{\alpha-1}}{\Gamma_q(\alpha)}+\frac{\mu}{M}H(\eta,qs))d_qs = \xi_2,$$

and

$$\Box Tu\Box_\chi \geq (\int_0^1|g(t,qs)f(s,u(s))|d_qs$$

$$+\frac{\mu t^2}{M}\int_0^1|H(\eta,qs)f(s,u(s))|d_qs)$$

$$\geq \frac{\mu t^2}{M}(\int_0^1 H(\eta,qs)d_qs)l\xi_1 = \xi_1.$$

Thus, we get $T:\Omega[\xi_1,\xi_2]\to\Omega[\xi_1,\xi_2]$.

Let $u_0(t) = \xi_2, 0\leq t\leq1$, then $u_0\in\Omega[\xi_1,\xi_2]$.

Let $u_1(t) = Tu_0(t)$, then $u_1\in\Omega[\xi_1,\xi_2]$. We denote

$$u_{m+1} = Tu_m = T^{m+1}u_0, m=0,1,2,\cdots$$

According to $T:\Omega[\xi_1,\xi_2]\to\Omega[\xi_1,\xi_2]$, we have

$$u_n\in\Omega[\xi_1,\xi_2], n=0,1,2,\cdots.$$

Since T is completely continuous, we assert that $\{u_m\}_{m=1}^\infty$ has a convergent subsequence $\{u_{m_k}\}_{k=1}^\infty$, there exists

$$u^*\in\Omega[\xi_1,\xi_2].$$

Such that $\lim_{k\to+\infty}u_{m_k} = u^*$.

Since $u_1\in\Omega[\xi_1,\xi_2]$, then

$$\xi_1\leq u_1\leq\|u_1\|\chi\leq\xi_2 = u_0.$$

According to the definition of T and (H_f), we have

$$u_1(t) = (Tu_0)(t) = \int_0^1 G(t,qs)f(s,u(s))d_qs$$

$$\leq\int_0^1\rho(s)f(s,\xi_2)d_qs$$

$$\leq L\xi_2\int_0^1\left(\frac{q^2(1-qs)^{\alpha-1}}{\Gamma_\alpha(\alpha)}+\frac{\mu}{M}H(\eta,qs)\right)d_qs = \xi_2 = u_0(t),$$

which implies $u_2(t) = Tu_1(t)\leq Tu_0 = u_1(t), 0\leq t\leq1$.

By introduction, we have $u_{m+1}(t)\leq u_m(t)$ for

$$0\leq t\leq1, m=0,1,2,\cdots$$

Thus, there exists $u^*\in\Omega[\xi_1,\xi_2]$ such that $\lim_{m\to\infty}u_m = u^*$.

From the continuity of T and $u_{m+1} = Tu_m$, we have

$$u^* = Tu^*.$$

The proof is completed.

Our next existence result is based on Krasnoselskii's fixed point theorem.

Lemma 3.2（Krasonselskii's）Let E be a Banach space, and let P$\subset$E be acone. Assume Ω1, Ω2 are open subsets of E with $\theta\in\Omega_1\subset\bar\Omega_1\subset\Omega_2$ and let $T:P\bigcap(\bar\Omega_2\backslash\Omega_1)\to P$ be a completely continuous operator such that

$$\|Tu\|\geq\|u\|, u\in P\cap\partial\Omega_1,\quad\|Tu\|\leq\|u\|, u\in P\cap\partial\Omega_2.$$

Then T has at least one fixed point in $P\cap(\bar\Omega_2\backslash\Omega_1)$.

Theorem 3.3 Let $f(t,u)$ be a nonnegative continuous function on [0,1]×R*.In addition, we assume that

(H1) There exists a positive constant r_1 such that

$$f(t,u)\geq kr_1,\quad(t,u)\in[\tau_3,\tau_4]\times[0,r_1],$$

where $\tau_3 = q^{m_3}, \tau_4 = q^{m_4}$ with $m_3,m_4\in N_0, m_3\geq m_4$,and

$$k\geq(\int_{\tau_3}^{\tau_4}(g(qs,qs)+\frac{\mu}{M}H(\eta,qs))d_qs)^{-1}$$

(H2) There exists a positive constant r_2 with $r_2 > r_1$ such that

$$f(t,u)\geq kr_1,(t,u)\in[0,1]\times[0,r_2],$$

Then the BVP (1,1) has at least one positive solution u_0 satisfying $0 < r_1\leq\|u_0\|x\leq r_2$.

Proof: By Lemma 2.8,we obtain that

$$\max_{0\leq t\leq1}g(t,qs) = g(qs,qs).$$

Let

$$\Omega_1 = \{u\in X:\|u\|x < r_1\}.$$

For any $u\in X\cap\partial\Omega_1$, according to (H1) and the definitions of τ_3 and τ_4, we obtain

$$\|Tu\|x = \max_{0\leq t\leq1}\int_0^1 g(t,qs)f(s,u(s))d_qs$$

$$+\max_{0\leq t\leq1}\int_0^1\frac{\mu t^2}{M}H(\eta,qs)f(s,u(s))d_qs$$

$$=\int_0^1\left[g(qs,qs)+\frac{\mu}{M}H(\eta,qs)\right]f(s,u(s))d_qs$$

$$\geq\int_{\tau_3}^{\tau_4}(g(qs,qs)+\frac{\mu}{M}H(\eta,qs))f(s,u(s))d_qs$$

$$\geq kr_1\int_{\tau_3}^{\tau_4}(g(qs,qs)+\frac{\mu}{M}H(\eta,qs))d_qs$$

$$\geq\|u\|x = r_1.$$

Let

$$\Omega_2 = \{u\in X:\|u\|x < r_2\}.$$

For any $u\in X\cap\partial\Omega_2$, by (H2) and Lemma 2.9, we have

$$\|Tu\|x = \max_{0\leq t\leq1}\int_0^1 G(t,qs)f(s,u(s))d_qs$$

$$\leq Lr_2\int_0^1\rho(s)d_qs$$

$$\le Lr_2 \int_0^1 (\frac{q^2(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{\mu}{M} H(\eta, qs)) d_q s$$

$$= \|u\| x = r_2.$$

Now, an application of Lemma 3.2 concludes the proof.

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