

On a new hypergeometric identity deducible by using Beta integral method

Medhat A. Rakha and Arjun K. Rathie

Abstract— The aim of this short research note is to provide a new and interesting hypergeometric identity by applying the well known beta integral method which was used successfully and systematically by Krattenthaler and Rao in their well known and very interesting research paper.

The identity is derived with the help of an extension of a quadratic transformation obtained very recently by the authors.

Keywords— Hypergeometric Series; Quadratic Transformation, Beta Integral

I. Introduction

We begin by recalling the following quadratic transformation formula due to Kummer [4]

$$(1-x)^{-2a} {}_2F_1 \left[\begin{matrix} 2a, & b; \\ 2b; \end{matrix} \middle| -\frac{2x}{1-x} \right] = {}_2F_1 \left[\begin{matrix} a, & a + \frac{1}{2}; \\ b + \frac{1}{2}; \end{matrix} \middle| x^2 \right]. \quad (1.1)$$

This result was independently re-discovered by Ramanujan [1, Entry 2, p. 49].

Using (1.1) and by applying the so-called beta integral method, recently Krattenthaler and Rao [3, Eq. (3.4), p. 64] obtained the following interesting identity

$${}_3F_2 \left[\begin{matrix} 2a, & b, & d; \\ 2b, & 1+2a+d-e; \end{matrix} \middle| 2 \right] = \frac{\Gamma(e-2a)\Gamma(e-d)}{\Gamma(e)\Gamma(e-2a-d)} \quad (1.2)$$

$$\times {}_4F_3 \left[\begin{matrix} a, & a + \frac{1}{2}, & \frac{1}{2}d, & \frac{1}{2}d + \frac{1}{2}; \\ b + \frac{1}{2}, & \frac{1}{2}e, & \frac{1}{2}e + \frac{1}{2}; \end{matrix} \middle| 1 \right]$$

provided $2a$ or b be a non-positive integer.

Recently, Kim, et al. [2] have given an interesting extension of Kummer's quadratic transformation (1.1) in the form

$$(1-x)^{-2a} {}_3F_2 \left[\begin{matrix} 2a, & b, & d+1; \\ 2b+1, & d; \end{matrix} \middle| -\frac{2x}{1-x} \right] = {}_2F_1 \left[\begin{matrix} a, & a + \frac{1}{2}; \\ b + \frac{1}{2}; \end{matrix} \middle| x^2 \right] + \frac{2a \left(1 - \frac{2b}{d}\right)}{(2b+1)} x {}_2F_1 \left[\begin{matrix} a, & a + \frac{1}{2}; \\ b + \frac{3}{2}; \end{matrix} \middle| x^2 \right]. \quad (1.3)$$

Very recently, another extension of the Kummer transformation (1.1) was given by Rakha, et al. [5] in the form

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$$\begin{aligned}
 & (1-x)^{-2a} {}_3F_2 \left[\begin{matrix} 2a, & b, & d+2; \\ 2b+2, & d; \end{matrix} \right. \left. -\frac{2x}{1-x} \right] \\
 & = {}_3F_2 \left[\begin{matrix} a, & a+\frac{1}{2}, & \frac{1}{4c}+1; \\ b+\frac{3}{2}, & \frac{1}{4c}; \end{matrix} \right. \left. x^2 \right] \\
 & + \frac{2a(d-2b)x}{d(b+1)} {}_2F_1 \left[\begin{matrix} a+\frac{1}{2}, & a+1; \\ b+\frac{3}{2}; \end{matrix} \right. \left. x^2 \right]
 \end{aligned} \tag{1.4}$$

where

$$c = \frac{\left(\frac{1-a}{2-d}\right)}{a+1} + \frac{a}{d(d+1)}, d \neq 0, -1, -2, \dots \tag{1.5}$$

The aim of this short research note is to provide a new and interesting hypergeometric identity by applying the so-called beta integral method to the extension of the quadratic transformation formula (1.3). The identity (1.2) obtained earlier by Krattenthaler and Rao follows special cases of our main findings.

II. A New Hypergeometric Identity

Our main identity to be proved in this short research note is given in the following theorem.

Theorem 2.1 *If at least one of $2a$, e or d is a non-positive integer, then the following identity holds true:*

$$\begin{aligned}
 & {}_4F_3 \left[\begin{matrix} 2a, & b, & d+1, & e; \\ 2b+1, & d, & 1+e+2a-f; \end{matrix} \right. \left. 2 \right] \\
 & = \frac{\Gamma(f-e)\Gamma(f-2a)}{\Gamma(f)\Gamma(f-e-2a)} \\
 & \times \left[{}_4F_3 \left[\begin{matrix} a, & a+\frac{1}{2}, & \frac{1}{2}e, & \frac{1}{2}e+\frac{1}{2}; \\ b+\frac{1}{2}, & \frac{1}{2}f, & \frac{1}{2}f+\frac{1}{2}; \end{matrix} \right. \right. \left. \left. 1 \right] \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{2ae\left(1-\frac{2b}{d}\right)}{f(2b+1)} \\
 & \times {}_4F_3 \left[\begin{matrix} a, & a+\frac{1}{2}, & \frac{1}{2}e+\frac{1}{2}, & \frac{1}{2}e+1; \\ b+\frac{3}{2}, & \frac{1}{2}f+\frac{1}{2}, & \frac{1}{2}f+1; \end{matrix} \right. \left. 1 \right] \tag{2.1}
 \end{aligned}$$

Proof. In order to establish the identity (2.1), we proceed as follows. Let us assume that $2a$ be a non-positive integer, say $2a = -m$, where m is a positive integer. Now multiply the left-hand side of (1.4) by $x^{e-1}(1-x)^{f-e-1}$, where we suppose temporarily that $Re(f) > Re(e) > 0$ and integrating the resulting equation with respect to x from 0 to 1 and denoting it by S_1 , we have

$$S_1 = \int_0^1 x^{e-1}(1-x)^{f-e-2a-1} {}_3F_2 \left[\begin{matrix} 2a, & b, & d+2; \\ 2b+2, & d; \end{matrix} \right. \left. -\frac{2x}{1-x} \right] dx.$$

Expressing ${}_3F_2$ as a series, changing the order of integration and summation (which is easily seen to be justified due to uniform convergence of the series involved in the process), we have

$$S_1 = \sum_{n=0}^{\infty} \frac{(2a)_n (b)_n (d+2)_n (-2)^n}{(2b+2)_n (d)_n n!} \int_0^1 x^{e+n-1} (1-x)^{f-e-2a-n-1} dx.$$

Evaluating the Beta integral and using the elementary identity

$$\Gamma(\alpha-n) = \frac{(-1)^n \Gamma(\alpha)}{(1-\alpha)_n},$$

we have after some algebra

$$\begin{aligned}
 S_1 & = \frac{\Gamma(e)\Gamma(f-e-2a)}{\Gamma(f-2a)} \\
 & \times \sum_{n=0}^{\infty} \frac{(2a)_n (b)_n (d+2)_n (e)_n}{(2b+2)_n (d)_n (1-f+e+2a)_n n!}
 \end{aligned}$$

summing up the inner series, we have

$$\begin{aligned}
 S_1 & = \frac{\Gamma(e)\Gamma(f-e-2a)}{\Gamma(f-2a)} \\
 & \times {}_4F_3 \left[\begin{matrix} 2a, & b, & d+2, & e; \\ 2b+2, & d, & 1-f+e+2a; \end{matrix} \right. \left. 2 \right] \tag{2.2}
 \end{aligned}$$

In the same manner, multiply the right-hand side of (1.4) by $x^{e-1}(1-x)^{f-e-1}$ and integrating the resulting equation with respect to x from 0 to 1 and denoting it by S_2 and then proceeding exactly as we have done above, we finally have after some simplification:

$$S_2 = \frac{\Gamma(e)\Gamma(f-e)}{\Gamma(f)} \times_5 F_4 \left[\begin{matrix} a, & a + \frac{1}{2}, & \frac{1}{4c} + 1, & \frac{1}{2}e, & \frac{1}{2}e + \frac{1}{2}; \\ b + \frac{3}{2}, & \frac{1}{4c}, & \frac{1}{2}f, & \frac{1}{2}f + \frac{1}{2}; \end{matrix} \right] + \frac{2a(d-2b)}{d(b+1)} \frac{\Gamma(e+1)\Gamma(f-e)}{\Gamma(f+1)} \times_4 F_3 \left[\begin{matrix} a + \frac{1}{2}, & a + 1, & \frac{1}{2}e + \frac{1}{2}, & \frac{1}{2}e + 1; \\ b + \frac{3}{2}, & \frac{1}{2}f + \frac{1}{2}, & \frac{1}{2}f + 1; \end{matrix} \right] \quad (2.3)$$

Finally, equating (2.2) and (2.3), we easily arrive at the desired result (2.1). Exactly the same process can be done in case of e or d . This completes the proof of our theorem. \square
 We conclude this section by remarking that the above restrictions on the parameters e and f may now be appeal to analytic continuation.

Corollary 2.2 *In our theorem, if we set $d = 2b$, then at once recover Krathenthaler and Rao result (1.2). Thus our result (2.1) may be regarded as an extension of (1.2).*

Remark: *For an interesting generalization of Krattenthaler and Rao identity (1.2), see a recent paper by Ibrahim, et. al. [6].*

Conflict of Interests

The authors declare that they have no conflict of interests.

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2. All authors contributed equally in this paper. They read and approved the final manuscript.

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$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \dots$$
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